

## FUNCTIONALS FOR MULTILINEAR FRACTIONAL EMBEDDING

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ABSTRACT. A novel representation is developed as a measure for multilinear fractional embedding. Corresponding extensions are given for the Bourgain-Brezis-Mironescu theorem and Pitt's inequality. New results are obtained for diagonal trace restriction on submanifolds as an application of the Hardy-Littlewood-Sobolev inequality. Smoothing estimates are used to provide new structural understanding for density functional theory, the Coulomb interaction energy and quantum mechanics of phase space. Intriguing connections are drawn that illustrate interplay among classical inequalities in Fourier analysis.

## 1. MULTILINEAR EMBEDDING

A problem of central interest for embedding is how to characterize the action of multilinear fractional smoothing: that is, control by the operator

$$\begin{aligned} (\Lambda_\alpha f)(x_1, \dots, x_m) &= \left[ \prod_{k=1}^m (-\Delta_k / 4\pi^2)^{\alpha_k/2} f \right] (x_1, \dots, x_m) \\ &= \mathcal{F}^{-1} \left[ \prod_k |\xi_k|^{\alpha_k} \widehat{f}(\xi_1, \dots, \xi_m) \right] (x_1, \dots, x_m) \end{aligned} \quad (1)$$

where  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $x_k \in \mathbb{R}^n$ ,  $\alpha = (\alpha_1, \dots, \alpha_m)$ ,  $|\alpha| = \sum \alpha_k$ ,  $0 < \alpha_k < n$ ,  $\Delta_k$  is the standard Laplacian on  $\mathbb{R}^n$  in the variable  $x_k$ , and

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int e^{2\pi i x \xi} f(x) dx .$$

Examples of how such control can be utilized are contained in ([7], [8], [9], [14]). Our objective here is to consider a corresponding functional suggested by the Aronszajn-Smith formula:

$$I_{p,\alpha}(f) = \int_{\mathbb{R}^{mn} \times \mathbb{R}^{mn}} \prod |x_k - y_k|^{-n-p\alpha_k} \left| \sum_{w=x,y} (-1)^{\sigma(y)} f(w_1, \dots, w_m) \right|^p dx dy \quad (2)$$

where  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $\alpha = (\alpha_1, \dots, \alpha_m)$ ,  $0 < \alpha_k < 1$ ,  $1 < p < n/\alpha_k$  for all  $k$ , and  $\sigma(y)$  counts the number of  $y$  values in the expression  $f(w_1, w_2, \dots, w_m)$  — for example,  $\sigma(y) = 3$  in the case  $f(x_1, y_2, y_3, x_4, y_5, x_6, \dots, x_m)$ . Related to this functional, one can give a non-local representation for multilinear fractional smoothness:

$$(\Lambda_\alpha f)(x_1, \dots, x_m) = \prod_k (2/D_{\alpha_k}) \int_{\mathbb{R}^{mn}} \prod |x_k - y_k|^{-n-\alpha_k} \left[ \sum_{w=x,y} (-1)^{\sigma(y)} f(w_1, \dots, w_m) \right] dy \quad (3)$$

Using the classical formula of Aronszajn-Smith and simple iteration:

**Lemma 1** (Multilinear Aronszajn-Smith Formula).

$$I_{2,\alpha}(f) = \left[ \prod D_{2\alpha_k} \right] \int_{\mathbb{R}^{mn}} |\Lambda_\alpha f|^2 dx \quad (4)$$

where

$$D_\beta = \frac{4}{\beta} \pi^{-n/2+\beta} \frac{\Gamma(1-\beta/2)}{\Gamma(\frac{n+\beta}{2})}$$

and  $0 < \alpha_k < \min\{1, n/2\}$ .

*Proof.* Apply the classical Aronszajn-Smith formula to successive variables:

$$\begin{aligned} & \int |\widehat{f}(\xi_1, \xi_2, \xi')|^2 |\xi_1|^{2\alpha_1} |\xi_2|^{2\alpha_2} d\xi_1 d\xi_2 \\ &= (D_{2\alpha_1})^{-1} \int \frac{|f(x_1, \xi_2, \xi') - f(y_1, \xi_2, \xi')|^2}{|x_1 - y_1|^{n+2\alpha_1}} |\xi_2|^{2\alpha_2} dx_1 dy_1 d\xi_2 \\ &= (D_{2\alpha_1} D_{2\alpha_2})^{-1} \int \frac{|[f(x_1, x_2, \xi') - f(y_1, x_2, \xi')] - [f(x_1, y_2, \xi') - f(y_1, y_2, \xi')]|^2}{|x_1 - y_1|^{n+2\alpha_1} |x_2 - y_2|^{n+2\alpha_2}} dx_1 dx_2 dy_1 dy_2 \end{aligned}$$

where  $\xi' = (\xi_3, \dots, \xi_m)$ . Continue this process until all the Fourier transform variables  $\xi_k$  are exhausted.  $\square$

Observe that if  $f(x_1, \dots, x_m) = g(x_1)h(x_2, \dots, x_m)$  then

$$\sum_{w=x,y} (-1)^{\sigma(y)} f(w_1, \dots, w_m) = [g(x_1) - g(y_1)] \sum_{w=x,y} (-1)^{\sigma(y)} h(w_2, \dots, w_m)$$

For product functions  $f(x_1, \dots, x_m) = \prod f_k(x_k)$

$$I_{p,\alpha}(f) = \prod_k \left[ \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-n-p\alpha_k} |f_k(x) - f_k(y)|^p dx dy \right]$$

This splitting, the utilization of iteration methods, and the product structure, suggests that the issue here is not a true multilinear problem, and that product functions will likely characterize results.

**Theorem 1** (Multilinear Pitt's Inequality). *Let  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $0 < \alpha_k < 1$  and  $1 \leq p < \min\{n/\alpha_k\}$ ; then*

$$I_{p,\alpha}(f) \geq \prod_k D_{p,\alpha_k} \int_{\mathbb{R}^{mn}} \prod |x_k|^{-p\alpha_k} |f(x)|^p dx \quad (5)$$

$$D_{p,\beta} = \int_{\mathbb{R}^n} |1 - |x|^{-\lambda}|^p |x - \eta|^{-n-p\beta} dx$$

for  $\lambda = (n - p\beta)/p$  and  $\eta \in S^{n-1}$ .

*Proof.* This result follows from successive application of Theorem 4.1 in [6] (see also Lemma 1 in [8]). Observe that

$$I_{p,\alpha}(f) \geq D_{p,\alpha_1} \int |x_1|^{-p\alpha_1} \prod_{k \geq 2} |x_k - y_k|^{-n-p\alpha_k} \left| \sum_{w=x,y} (-1)^{\sigma(y)} f(x_1, w_2, \dots, w_m) \right|^p dx dy ;$$

continue this argument for the variables  $x_k, y_k$  for  $k \geq 2$  to obtain the full inequality (5). The constant is sharp as can be seen from the calculation for product functions.  $\square$

For  $p = 2$ , a more explicit realization can be given for the constant (see discussion of Pitt's inequality in [5], [6]):

**Corollary.** For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $0 < \alpha_k < 1$

$$I_{2,\alpha}(f) \geq \prod_k \left[ D_{2\alpha_k} / C_{2\alpha_k} \right] \int_{\mathbb{R}^{mn}} \prod |x_k|^{-2\alpha_k} |f(x)|^2 dx \quad (6)$$

$$C_\beta = \pi^\beta \left[ \Gamma\left(\frac{n-\beta}{4}\right) / \Gamma\left(\frac{n+\beta}{4}\right) \right]^2$$

**Theorem 2** (Multilinear Bourgain-Brezis-Mironescu). For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $0 < \beta < 1$ ,  $1 \leq p < n/\beta$  and  $\alpha_k = \beta$  for all  $k$ ;

$$I_{p,\alpha}(f) \geq (c_{n,p})^m \left( \int_{\mathbb{R}^{mn}} |f|^{q^*} dx \right)^{p/q^*}, \quad q^* = \frac{pn}{n-p\beta} \quad (7)$$

where  $c_{n,p}$  is the optimal embedding constant on  $\mathbb{R}^n$ .

**Corollary.** For  $p = 2 < n/\beta$ , the value of  $c_{n,2}$  is given by

$$c_{n,2} = \frac{2}{\beta} \frac{\Gamma(1-\beta)}{\Gamma(\frac{n}{2}-\beta)} \left[ \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right]^{2\beta/n} \quad (8)$$

*Proof.* This multilinear embedding result is obtained by applying the Bourgain-Brezis-Mironescu theorem in the context of a multiplicative iteration scheme with the aid of the Minkowski inequality for integrals:

$$\begin{aligned} & \int_{\mathbb{R}^{mn} \times \mathbb{R}^{mn}} \prod |x_k - y_k|^{-n-p\alpha} \left| \sum_{w=x,y} (-1)^\sigma f(w) \right|^p dx dy \\ & \geq c_{n,p} \int_{\mathbb{R}^{(m-1)n} \times \mathbb{R}^{(m-1)n}} \prod |x'_k - y'_k|^{-n-p\alpha} \left[ \int_{\mathbb{R}^n} \left| \sum_{w=x',y'} (-1)^\sigma f(v_1, w) \right|^{q^*} dv_1 \right]^{p/q^*} dx' dy' \\ & \geq c_{n,p} \left[ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^{(m-1)n} \times \mathbb{R}^{(m-1)n}} \prod |x'_k - y'_k|^{-n-p\alpha} \left| \sum_{w=x',y'} (-1)^\sigma f(v_1, w) \right|^p dx' dy' \right]^{q^*/p} dv_1 \right]^{p/q^*} \\ & \geq (c_{n,p})^2 \left[ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^{(m-2)n} \times \mathbb{R}^{(m-2)n}} \prod |x''_k - y''_k|^{-n-p\alpha} \right. \right. \\ & \quad \left. \left[ \int_{\mathbb{R}^n} \left| \sum_{w=x'',y''} (-1)^\sigma f(v_1, v_2, w) \right|^{q^*} dv_2 \right]^{p/q^*} dx'' dy'' \right]^{q^*/p} dv_1 \right]^{p/q^*} \\ & \geq (c_{n,p})^2 \left[ \int_{\mathbb{R}^n \times \mathbb{R}^n} \left[ \int_{\mathbb{R}^{(m-2)n} \times \mathbb{R}^{(m-2)n}} \prod |x''_k - y''_k|^{-n-p\alpha} \right. \right. \\ & \quad \left. \left| \sum_{w=x'',y''} (-1)^\sigma f(v_1, v_2, w) \right|^p dx'' dy'' \right]^{q^*/p} dv_1 dv_2 \right]^{p/q^*} \\ & \geq \dots = (c_{n,p})^m \left[ \int_{\mathbb{R}^{mn}} |f(v)|^{q^*} dv \right]^{p/q^*} \end{aligned}$$

Here primes denote:  $x' = (x_2, \dots, x_m)$  and  $x'' = (x_3, \dots, x_m)$ . The first inequality follows from application of the Bourgain-Brezis-Mironescu theorem on  $\mathbb{R}^n$ ; the second inequality invokes

Minkowski's inequality for integrals in the form

$$\int \left[ \int |h|^q d\mu \right]^{p/q} d\nu \geq \left[ \int \left[ \int |h|^p d\nu \right]^{q/p} d\mu \right]^{p/q}, \quad q > p.$$

The sharpness of the constant is demonstrated by using product functions —  $f(x) = \prod f_k(x_k)$ . The sharp  $L^2$  embedding constant  $c_{n,2}$  was first noted in [6] (see Theorem 3.3 on page 187).  $\square$

## 2. DIAGONAL TRACE RESTRICTION

The objective here is to develop an overall framework for the structure of multilinear convolution operators and the representation of the Hardy-Littlewood-Sobolev inequality from the perspective defined by multilinear Sobolev embedding. To enable a better understanding for the role of geometric symmetry and the application of duality arguments, diagonal trace restriction is considered in the context of a lower-dimensional manifold — namely, the unit sphere. This approach extends the structure of classical trace inequalities from harmonic extension of boundary values (for the upper half-space or the interior of the unit ball, see [1], [16]) to restriction phenomena on surfaces with curvature. Questions about restriction for the Fourier transform on manifolds with curvature and Strichartz inequalities involve greater depth and subtlety as illustrated by the original Stein-Tomas inequality.

Determination of sharp constants for diagonal trace restriction estimates was initiated in [7] and extended in [9]. Motivated by the principal results from [7] (Theorems 1 and 2), restriction estimates are obtained here for the sphere  $S^{n-1}$ . First consider the basic estimates

**Pitt's inequality.** ( $n - \beta = mn - 2\alpha$ ,  $\alpha = \sum \alpha_k$ ,  $0 < \beta < n$ )

$$\int_{\mathbb{R}^n} |x|^{-\beta} |f(\underbrace{x, \dots, x}_{m \text{ slots}})|^2 dx \leq C_\beta \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Lambda_\alpha f|^2 dx \quad (9)$$

$$C_\beta = \pi^{-(m-1)n/2 + 2\alpha} \prod_{k=1}^m \left[ \frac{\Gamma(\frac{n}{2} - \alpha_k)}{\Gamma(\alpha_k)} \right] \left[ \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{n-\beta}{2})} \right] \left[ \frac{\Gamma(\frac{n-\beta}{4})}{\Gamma(\frac{n+\beta}{4})} \right]^2$$

**Hardy-Littlewood-Sobolev inequality.** ( $mn - 2\alpha = 2n/q$ )

$$\left[ \int_{\mathbb{R}^n} |f(\underbrace{x, \dots, x}_{m \text{ slots}})|^q dx \right]^{2/q} \leq F_\alpha \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Lambda_\alpha f|^2 dx \quad (10)$$

$$F_\alpha = \pi^\alpha \prod_{k=1}^m \left[ \frac{\Gamma(\frac{n}{2} - \alpha_k)}{\Gamma(\alpha_k)} \right] \left[ \frac{\Gamma(\alpha - (m-1)n/2)}{\Gamma(\alpha - (m-2)n/2)} \right] \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{\frac{2\alpha - (m-1)n}{n}}$$

Motivated by the proof of the Hardy-Littlewood-Sobolev inequality in [7], a diagonal trace restriction inequality can be given in terms of the  $(n-1)$  dimensional unit sphere. The proof uses duality and a reduction to the  $(n-1)$  dimensional Hardy-Littlewood-Sobolev inequality on the sphere.

**Theorem 3** (Multilinear Hardy-Littlewood-Sobolev). *For  $f \in \mathcal{S}(\mathbb{R}^{mn})$  and  $mn - 2\alpha = 2(n - 1)/q$ ,  $q > 2$ ,  $n > 1$*

$$\left[ \int_{S^{n-1}} |f(\underbrace{\xi, \dots, \xi}_{m \text{ slots}})|^q d\xi \right]^{2/q} \leq K_\alpha \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Lambda_\alpha f|^2 dx \quad (11)$$

$$K_\alpha = (2\pi)^{2\alpha} (4\pi)^{-mn/2} \prod_{k=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_k)}{\Gamma(\alpha_k)} \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \frac{\Gamma[(n-1)(\frac{1}{2} - \frac{1}{q})]}{\Gamma(\frac{n-1}{p})}$$

Here  $d\xi$  denotes normalized surface measure on the sphere  $S^{n-1}$ .

*Proof.* Inequality (11) is equivalent to the multilinear fractional integral inequality:

$$\begin{aligned} & \left[ \int_{S^{n-1}} \left| \int_{\mathbb{R}^{mn}} \prod_{k=1}^m |\xi - y_k|^{-(n-\alpha_k)} f(y_1, \dots, y_m) dy \right|^q d\xi \right]^{2/q} \\ & \leq G_\alpha \int_{\mathbb{R}^{mn}} |f(x_1, \dots, x_m)|^2 dx \end{aligned} \quad (12)$$

$$K_\alpha = \pi^{-mn+2\alpha} \prod_{k=1}^m \left[ \frac{\Gamma(\frac{n-\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})} \right]^2 G_\alpha$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{mn}} \left| \int_{S^{n-1}} \prod_{k=1}^m |y_k - \xi|^{-(n-\alpha_k)} g(\xi) d\xi \right|^2 dy \leq G_\alpha \left[ \int_{S^{n-1}} |g(\xi)|^p d\xi \right]^{2/p}$$

where  $1/p + 1/q = 1$ ,  $1 < p < 2$  and  $mn - 2\alpha = 2(n - 1)/q$ . The left-hand side now becomes

$$\int_{S^{n-1} \times S^{n-1} \times \mathbb{R}^{mn}} g(\xi) \prod_{k=1}^m |y_k - \xi|^{-(n-\alpha_k)} \prod_{k=1}^m |y_k - \eta|^{-(n-\alpha_k)} g(\eta) d\xi d\eta dy$$

Integrating out the  $y_k$  variables

$$\int_{S^{n-1} \times S^{n-1}} g(\xi) |\xi - \eta|^{-mn+2\alpha} g(\eta) d\xi d\eta \leq H_\alpha \left[ \int_{S^{n-1}} |g(\xi)|^p d\xi \right]^{2/p}$$

$$K_\alpha = \pi^{-mn/2+2\alpha} \prod_{k=1}^m \Gamma\left(\frac{n}{2} - \alpha_k\right) / \Gamma(\alpha_k) H_\alpha$$

Since  $mn - 2\alpha = 2(n - 1)/q$ , this becomes the classical Hardy-Littlewood-Sobolev inequality on the  $(n - 1)$  dimensional sphere  $S^{n-1}$ :

$$\int_{S^{n-1} \times S^{n-1}} g(\xi) |\xi - \eta|^{-2(n-1)/q} g(\eta) d\xi d\eta \leq H_\alpha \left[ \int_{S^{n-1}} |g(\xi)|^p d\xi \right]^{2/p}$$

$$H_\alpha = 2^{-2(n-1)/q} \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \frac{\Gamma[(n-1)(\frac{1}{2} - \frac{1}{q})]}{\Gamma(\frac{n-1}{p})}$$

Then

$$K_\alpha = (2\pi)^{2\alpha} (4\pi)^{-mn/2} \prod_{k=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_k)}{\Gamma(\alpha_k)} \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \frac{\Gamma[(n-1)(\frac{1}{2} - \frac{1}{q})]}{\Gamma(\frac{n-1}{p})}.$$

Extremal functions for (11) and (12) are determined up to conformal automorphism on the sphere  $S^{n-1}$  as equivalent to

$$\int_{S^{n-1}} \prod_{k=1}^m |x_k - \xi|^{-(n-\alpha_k)} d\xi .$$

□

Observe that the duality argument used in this proof provides the following restriction result for a spherical surface as determined by fractional smoothness. This result was obtained earlier by Bez, Machihara and Sugimoto (personal communication — see [11]). From the representation of the Hardy-Littlewood-Sobolev inequality as a smoothing estimate, one expects similar estimates to hold for any conformally equivalent setting (see equation (10) above and section 2 in [1]).

**Theorem 4.** *For  $f \in \mathcal{S}(\mathbb{R}^n)$  and  $n - 2\alpha = 2(n - 1)/q$  with  $q > 2$ ,  $n > 1$*

$$\left[ \int_{S^{n-1}} |f(\xi)|^q d\xi \right]^{2/q} \leq B_\alpha \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/2} f \right|^2 dx \quad (13)$$

$$B_\alpha = (2\pi)^{2\alpha} (4\pi)^{-n/2} \frac{\Gamma(\frac{n-1}{q})}{\Gamma(\frac{n-1}{p})} \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \frac{\Gamma[(n-1)(\frac{1}{2} - \frac{1}{q})]}{\Gamma[(n-1)(\frac{1}{2} - \frac{1}{q}) + \frac{1}{2}]}$$

*Proof.* This inequality corresponds to the special case  $m = 1$  in the previous argument and demonstrates that such estimates additionally hold for spherical restriction for all positive indices below the critical index  $q = 2(n - 1)/(n - 2\alpha)$ . □

To gain a better sense of the contrast for this trace estimate between harmonic extension of boundary values and global embedding, set  $\alpha = 1$  and raise the dimension by one so that critical index is given by  $q = 2n/(n - 1)$  for  $n > 1$ ; then

$$\left( \int_{S^n} |f(\xi)|^q d\xi \right)^{2/q} \leq b_n \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx \quad (14)$$

$$b_n = \frac{1}{4} \pi^{-(n+1)/2} \Gamma\left(\frac{n-1}{2}\right) .$$

On the other hand, using Theorem 4 from [1] for the critical index  $q = 2n/(n - 1)$

$$\left( \int_{S^n} |F(\xi)|^q d\xi \right)^{2/q} \leq 2b_n \int_{|x| \leq 1} |\nabla u|^2 dx + \int_{S^n} |F(\xi)|^2 d\xi \quad (15)$$

where  $u$  is the harmonic extension of  $F$  to the interior of the unit ball in  $\mathbb{R}^{n+1}$ . Both inequalities are sharp, and the doubling factor seems natural in view of symmetry. A precise derivation of the relation between the two inequalities using symmetrization is given in the Appendix.

The possibility of considering a spherical trace diagonal restriction corresponding to Pitt's inequality is less natural because the estimate is taken over a compact domain, and the critical index for embedding is not used. The nature of Pitt's inequality depends on the dilation character of the smoothing operator which will not play a new role for restriction on a compact manifold. Moreover, in contrast to the non-compact setting where extremals do not exist, one expects that in the compact case extremals are likely to exist.

## 3. DIAGONAL TRACE RESTRICTION ON SUBMANIFOLDS

Trace restriction from either the vantage point of harmonic extension or understanding models for many-body dynamics using the Gross-Pitaevskii hierarchy of density matrices seems naturally associated with control determined by multilinear fractional Sobolev embedding. But diagonal trace restriction on submanifolds is more directly a consequence of the Hardy-Littlewood-Sobolev embedding estimates, including more general formulations. First, a very general principle is outlined, and then explicit applications are developed including the case of flat submanifolds.

**Hardy-Littlewood-Sobolev principle — submanifold restriction.** For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $K$  a smooth submanifold of  $\mathbb{R}^n$ ,  $\sigma$  denotes a surface measure on  $K$ , and the index  $q$  depends on  $\alpha$  and  $K$ ; then

$$\left[ \int_{K \times \dots \times K} |f(\underbrace{w, \dots, w}_{m \text{ slots}})|^q d\sigma \right]^{2/q} \leq C_\alpha \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Lambda_\alpha f|^2 dx \quad (16)$$

This result is determined by the corresponding Hardy-Littlewood-Sobolev inequality on  $K$ :

$$\left| \int_{K \times K} g(u) |u - v|^{-\lambda} h(v) d\sigma d\sigma \right| \leq C'_\alpha \|g\|_{L^p(K)} \|h\|_{L^p(K)} \quad (17)$$

where  $p$  is the dual exponent to  $q$  and  $\lambda = mn - 2\alpha$ ,  $\alpha = \sum \alpha_k$ .

The classical sense of trace operator is associated with harmonic extension and solutions of differential equations. But here consideration of diagonal trace restriction suggests a broader mechanism that couples fractional Sobolev embedding with estimates for multilinear potential operators and application of the Hardy-Littlewood-Sobolev inequality to obtain optimal bounds. Without being exhaustive, examples are given to suggest the range of results that may be obtained using diagonal trace restriction on submanifolds, including both flat and product submanifolds.

**Theorem 5** (flat submanifolds). For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $n = k + \ell$ ,  $1 \leq k, \ell$  and  $\bar{x} = (x, y)$  for  $x \in \mathbb{R}^k$  and  $y$  a fixed point in  $\mathbb{R}^\ell$  with  $mn - 2\alpha = 2k/q$ ,  $q > 2$ :

$$\left[ \int_{\mathbb{R}^k} |f(\underbrace{\bar{x}, \dots, \bar{x}}_{m \text{ slots}})|^q dx \right]^{2/q} \leq A_\alpha \int_{\mathbb{R}^{mn}} |\Lambda_\alpha f|^2 dx_1 \dots dx_m \quad (18)$$

$$A_\alpha = \pi^\alpha \frac{\Gamma[k(\frac{1}{p} - \frac{1}{2})]}{\Gamma(\frac{k}{p})} \left[ \frac{\Gamma(\frac{k}{2})}{\Gamma(k)} \right]^{1-2/p} \prod_{\ell=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_\ell)}{\Gamma(\alpha_\ell)}.$$

**Theorem 6** (Pitt's inequality). For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $n = k + \ell$ ,  $1 \leq k, \ell$  and  $\bar{x} = (x, y)$  for  $x \in \mathbb{R}^k$  and  $y$  a fixed point in  $\mathbb{R}^\ell$  with  $mn - 2\alpha = k - \beta$ ,  $0 < \beta < k$ :

$$\int_{\mathbb{R}^k} |x|^{-\beta} |f(\underbrace{\bar{x}, \dots, \bar{x}}_{m \text{ slots}})|^2 dx \leq C_{\beta,k} \int_{\mathbb{R}^n \times \dots \times \mathbb{R}^n} |\Lambda_\alpha f|^2 dx_1 \dots dx_m \quad (19)$$

$$C_{\beta,k} = \pi^{(-mn+k)/2+2\alpha} \prod_{j=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_j)}{\Gamma(\alpha_j)} \left[ \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{k-\beta}{2})} \right] \left[ \frac{\Gamma(\frac{k-\beta}{4})}{\Gamma(\frac{k+\beta}{4})} \right]^2$$

**Theorem 7** (product of spheres). For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $n = k + \ell + 2$ ,  $1 \leq k, \ell$  and  $\bar{\xi} = (\xi, \eta)$ ,  $\xi \in S^k$ ,  $\eta \in S^\ell$  with  $d\xi d\eta$  denoting normalized surface measure on  $S^k \times S^\ell$  with  $mn - 2\alpha = 2(k + \ell)/q$ ,

$q > 2$ :

$$\left[ \int_{S^k \times S^\ell} |f(\underbrace{\bar{\xi}, \dots, \bar{\xi}}_{m \text{ slots}})|^q d\xi d\eta \right]^{2/q} \leq B_{\alpha,k} \int_{\mathbb{R}^{mn}} |\Lambda_\alpha f|^2 dx \quad (20)$$

$$B_{\alpha,k} = (4\pi)^{-(k+\ell)/q} \pi^\alpha \prod_{j=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_j)}{\Gamma(\alpha_j)} \frac{\Gamma(k)\Gamma(\ell)}{\Gamma(k/2)\Gamma(\ell/2)} \frac{\Gamma[k(\frac{1}{2} - \frac{1}{q})]\Gamma[\ell(\frac{1}{2} - \frac{1}{q})]}{\Gamma(k/p)\Gamma(\ell/p)}$$

*Proof of Theorem 5.* Inequality (18) is equivalent to the multilinear fractional integral inequality:

$$\left[ \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^{mn}} \prod_{\ell=1}^m |\bar{x} - u_\ell|^{-(n-\alpha_\ell)} f(u_1, \dots, u_m) du \right|^q dx \right]^{2/q}$$

$$\leq A_{\alpha,1} \int_{\mathbb{R}^{mn}} |f(u_1, \dots, u_m)|^2 du$$

$$A_\alpha = \pi^{-mn+2\alpha} \prod_{\ell=1}^m \left[ \frac{\Gamma(\frac{n-\alpha_\ell}{2})}{\Gamma(\frac{\alpha_\ell}{2})} \right]^2 A_{\alpha,1}$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{mn}} \left| \int_{\mathbb{R}^k} \prod_{\ell=1}^m |u_\ell - \bar{x}|^{-(n-\alpha_\ell)} g(x) dx \right|^2 du \leq A_{\alpha,1} \left[ \int_{\mathbb{R}^k} |g(x)|^p dx \right]^{2/p}$$

where  $1/p + 1/q = 1$ ,  $1 < p < 2$  and  $mn - 2\alpha = 2k/q$ . The left-hand side now becomes

$$\int_{\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{mn}} g(x) \prod_{\ell=1}^m |u_\ell - \bar{x}|^{-(n-\alpha_\ell)} \prod_{\ell=1}^m |u_\ell - \bar{w}|^{-(n-\alpha_\ell)} g(w) dx dw du$$

where  $\bar{x} = (x, y)$   $\bar{w} = (w, y)$  with  $y$  is a fixed point in  $\mathbb{R}^{n-k}$ . But now this form is independent of the fixed point  $y$ . Integrating out the  $u_k$  variables

$$\int_{\mathbb{R}^k \times \mathbb{R}^k} g(x) |x - w|^{-mn+2\alpha} g(w) dx dw \leq A_{\alpha,2} \left[ \int_{\mathbb{R}^k} |g(x)|^p dx \right]^{2/p}$$

$$A_\alpha = \pi^{-mn/2+2\alpha} \prod_{\ell=1}^m \Gamma\left(\frac{n}{2} - \alpha_\ell\right) / \Gamma(\alpha_\ell) A_{\alpha,2}$$

Since  $mn - 2\alpha = 2k/q$ , this estimate becomes the classical Hardy-Littlewood-Sobolev inequality on the  $\mathbb{R}_k$ :

$$\int_{\mathbb{R}^k \times \mathbb{R}^k} g(w) |x - w|^{-2k/q} g(w) dx dw \leq A_{\alpha,2} \left[ \int_{\mathbb{R}^k} |g(x)|^p dx \right]^{2/p}$$

$$A_{\alpha,2} = \pi^{k/q} \frac{\Gamma[k(\frac{1}{p} - \frac{1}{2})]}{\Gamma(\frac{k}{p})} \left[ \frac{\Gamma(\frac{k}{2})}{\Gamma(k)} \right]^{1-2/p}$$

Then

$$A_\alpha = \pi^\alpha \frac{\Gamma[k(\frac{1}{p} - \frac{1}{2})]}{\Gamma(\frac{k}{p})} \left[ \frac{\Gamma(\frac{k}{2})}{\Gamma(k)} \right]^{1-2/p} \prod_{\ell=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_\ell)}{\Gamma(\alpha_\ell)}.$$



□

*Proof of Theorem 6.* Inequality (19) is equivalent to the multilinear fractional integral inequality:

$$\begin{aligned} & \int_{\mathbb{R}^k} |x|^{-\beta} \left| \int_{\mathbb{R}^{mn}} \prod_{j=1}^m |\bar{x} - u_j|^{-(n-\alpha_j)} f(u_1, \dots, u_m) du \right|^2 dx \\ & \leq E_{\alpha,1} \int_{\mathbb{R}^{mn}} |f(u_1, \dots, u_m)|^2 dx \\ & C_{\beta,k} = \pi^{-mn+2\alpha} \prod_{j=1}^m \left[ \frac{\Gamma(\frac{n-\alpha_j}{2})}{\Gamma(\frac{\alpha_j}{2})} \right]^2 E_{\alpha_1} \end{aligned}$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{mn}} \left| \int_{\mathbb{R}^k} \prod_{j=1}^m |u_j - \bar{x}|^{-(n-\alpha_j)} |x|^{-\beta/2} g(x) dx \right|^2 du \leq E_{\alpha,1} \int_{\mathbb{R}^k} |g|^2 dx$$

where  $mn - 2\alpha = k - \beta$ . The left-hand side now becomes

$$\int_{\mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{mn}} g(x) |x|^{-\beta/2} \prod_{j=1}^m |u_j - \bar{x}|^{-(n-\alpha_j)} \prod_{j=1}^m |u_j - \bar{w}|^{-(n-\alpha_j)} |w|^{-\beta/2} g(w) dx dw du$$

where  $\bar{x} = (x, y)$ ,  $\bar{w} = (w, y)$  with  $y$  a fixed point in  $\mathbb{R}^{n-k}$ . But now this form is independent of the fixed point  $y$ . Integrating out the  $u_j$  variables

$$\begin{aligned} & \int_{\mathbb{R}^k \times \mathbb{R}^k} g(x) |x|^{-\beta/2} |x - w|^{-mn+2\alpha} |w|^{-\beta/2} g(w) dx dw \leq E_{\alpha,2} \int_{\mathbb{R}^k} |g|^2 dx \\ & C_{\beta,k} = \pi^{-mn/2+2\alpha} \prod_{j=1}^m \Gamma\left(\frac{n}{2} - \alpha_j\right) / \Gamma(\alpha_j) E_{\alpha,2} \end{aligned}$$

Since  $mn - 2\alpha = k - \beta$ , this estimate becomes the classical Pitt's inequality on  $\mathbb{R}^k$ :

$$\begin{aligned} & \int_{\mathbb{R}^k \times \mathbb{R}^k} g(x) |x|^{-\beta/2} |x - w|^{-(k-\beta)} |w|^{-\beta/2} dx dw \leq E_{\alpha,2} \int_{\mathbb{R}^k} |g|^2 dx \\ & E_{\alpha,2} = \pi^{k/2} \left[ \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{k-\beta}{2})} \right] \left[ \frac{\Gamma(\frac{k-\beta}{4})}{\Gamma(\frac{k+\beta}{4})} \right]^2 \end{aligned}$$

Then

$$C_{\beta,k} = \pi^{(-mn+k)/2+2\alpha} \prod_{j=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_j)}{\Gamma(\alpha_j)} \left[ \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{k-\beta}{2})} \right] \left[ \frac{\Gamma(\frac{k-\beta}{4})}{\Gamma(\frac{k+\beta}{4})} \right]^2$$

□

*Proof of Theorem 7.* Inequality (20) is equivalent to the multilinear fractional inequality where  $\hat{x} = (\xi, \eta) \in S^k \times S^\ell$ :

$$\begin{aligned} & \left[ \int_{S^k \times S^\ell} \left| \int_{\mathbb{R}^{mn}} \prod_{j=1}^m |\hat{x} - y_j|^{-(n-\alpha_k)} f(y_1, \dots, y_m) dy \right|^q d\xi d\eta \right]^{2/q} \\ & \leq F_{\alpha,1} \int_{\mathbb{R}^{mn}} |f(x_1, \dots, x_m)|^2 dx \\ & B_{\alpha,k} = \pi^{-mn+2\alpha} \prod_{k=1}^m \left[ \frac{\Gamma(\frac{n-\alpha_k}{2})}{\Gamma(\frac{\alpha_k}{2})} \right]^2 F_{\alpha,1} \end{aligned}$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{mn}} \left| \int_{S^k \times S^\ell} \prod_{j=1}^m |u_j - \hat{x}|^{-(n-\alpha_j)} g(\hat{x}) d\xi d\eta \right|^2 du \leq F_{\alpha,1} \left[ \int_{S^k \times S^\ell} |g(\hat{x})|^p d\xi d\eta \right]^{2/p}$$

where  $1/p + 1/q = 1$ ,  $1 < p < 2$  and  $mn - 2\alpha = 2(k + \ell)/q$ . The left-hand side now becomes

$$\int_{M \times M \times \mathbb{R}^{mn}} g(\bar{x}) \prod_{j=1}^m |u_j - \bar{x}|^{-(n-\alpha_j)} \prod_{j=1}^m |u_j - \bar{w}|^{-(n-\alpha_j)} g(\bar{w}) d\bar{x} d\bar{w} du$$

where  $M = S^k \times S^\ell$ ,  $\bar{x} = (\xi, \eta)$ ,  $\bar{w} = (\xi', \eta')$ . Integrating out the  $u_k$  variables

$$\int_{M \times M} g(\bar{x}) \left[ |\xi - \xi'|^2 + |\eta - \eta'|^2 \right]^{-mn/2 + \alpha} g(\bar{w}) d\bar{x} d\bar{w} \leq F_{\alpha,2} \left[ \int_M |g|^p d\bar{x} \right]^{2/p}$$

$$B_{\alpha,k} = \pi^{-mn/2 + 2\alpha} \prod_{j=1}^m \Gamma\left(\frac{n}{2} - \alpha_j\right) / \Gamma(\alpha_j) F_{\alpha,2}$$

Since  $mn - 2\alpha = 2(k + \ell)/q$  and  $[|\xi - \xi'|^2 + |\eta - \eta'|^2]^{k+\ell} \geq |\xi - \xi'|^{2k} |\eta - \eta'|^{2\ell}$

$$\begin{aligned} & \int_{M \times M} g(\xi, \eta) \left[ |\xi - \xi'|^2 + |\eta - \eta'|^2 \right]^{-(k+\ell)/q} g(\xi', \eta') d\xi d\eta d\xi' d\eta' \\ & \leq \int_{M \times M} g(\xi, \eta) |\xi - \xi'|^{2k/q} |\eta - \eta'|^{-2\ell/q} g(\xi', \eta') d\xi d\eta d\xi' d\eta' \\ & \leq F_{\alpha,3} \left[ \int_M |g(\xi, \eta)|^p d\xi d\eta \right]^{2/p}. \end{aligned}$$

By using successive applications of the Hardy-Littlewood-Sobolev inequality on spheres

$$F_{\alpha,3} = 2^{-(k+\ell)/q} \frac{\Gamma(k)\Gamma(\ell)}{\Gamma(k/2)\Gamma(\ell/2)} \frac{\Gamma[k(\frac{1}{2} - \frac{1}{q})]\Gamma[\ell(\frac{1}{2} - \frac{1}{q})]}{\Gamma(k/p)\Gamma(\ell/p)}$$

Since  $F_{\alpha,3} > F_{\alpha,2}$ , a non-sharp value of  $B_{\alpha,k}$  is given by:

$$B_{\alpha,k} = (4\pi)^{-(k+\ell)/q} \pi^\alpha \prod_{j=1}^m \frac{\Gamma(\frac{n}{2} - \alpha_j)}{\Gamma(\alpha_j)} \frac{\Gamma(k)\Gamma(\ell)}{\Gamma(k/2)\Gamma(\ell/2)} \frac{\Gamma[k(\frac{1}{2} - \frac{1}{q})]\Gamma[\ell(\frac{1}{2} - \frac{1}{q})]}{\Gamma(k/p)\Gamma(\ell/p)}$$

□

In contrast to Theorems 5 and 6 where the constants are sharp, the resulting constant  $B_{\alpha,k}$  obtained here for Theorem 7 is not sharp because of using the geometric mean estimate. Further, embedding restriction for a product submanifold of spheres allows the embedding index  $q$  to decrease, that is to be closer to the index 2.

As observed above for the sphere, Theorem 5 will determine a restriction result for a subspace that includes the usual estimates for harmonic extension to a half-space.

**Theorem 8.** *For  $f \in \mathcal{S}(\mathbb{R}^n)$  and  $\bar{x} = (x, 0)$  with  $x \in \mathbb{R}^{n-1}$  with  $n - 2\alpha = 2(n - 1)/q$ ,  $q > 2$  ( $2\alpha - 1 > 0$ )*

$$\left[ \int_{\mathbb{R}^{n-1}} |f(\bar{x})|^q dx \right]^{2/q} \leq A_\alpha \int_{\mathbb{R}^n} |\Lambda_\alpha f|^2 dx \quad (21)$$

$$A_\alpha = \pi^\alpha \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\frac{n}{2} + \alpha - 1)} \frac{\Gamma(\frac{n}{2} - \alpha)}{\Gamma(\alpha)} \left[ \frac{\Gamma(n-1)}{\Gamma(\frac{n-1}{2})} \right]^{\frac{2\alpha-1}{n-1}}$$

*Proof.* This estimate corresponds to the case  $m = 1$  in Theorem 5.  $\square$

To match this result to harmonic extension, set  $\alpha = 1$  and raise the dimension by one so that the critical index is given by  $q = 2n/(n - 1)$ ; then for  $(x, y) \in \mathbb{R}^{n+1}$

$$\left( \int_{\mathbb{R}^n} |f(x, 0)|^q dx \right)^{2/q} \leq c_n \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx dy \quad (22)$$

$$c_n = \frac{1}{\sqrt{4\pi}} \frac{1}{n-1} \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{1/n}$$

But this inequality determines the classic result for harmonic extension on a half-space (see inequality 32 on page 231 in [1]):

$$\left( \int_{\mathbb{R}^n} |f(x)|^q dx \right)^{2/q} \leq 2c_n \int_{\mathbb{R}_+^{n+1}} |\nabla u|^2 dx dy \quad (23)$$

where  $n > 1$  and  $u$  is the harmonic extension of  $f$  to the upper half-space.

#### 4. FRACTIONAL EMBEDDING ON THE SPHERE

The emergence of restriction smoothing estimates on the sphere suggests that embedding estimates for fractional smoothness can also be obtained for the sphere in the form of an Aronszajn-Smith formula using spherical harmonics and a Bourgain-Brezis-Mironescu theorem.

**Lemma 2** (après Aronszajn-Smith). *Let  $F = \sum Y_k$  where  $Y_k$  is a spherical harmonic of degree  $k$ ; then for  $0 < \beta < \min\{1, n/2\}$*

$$\int_{S^n \times S^n} \frac{|F(\xi) - F(\eta)|^2}{|\xi - \eta|^{n+2\beta}} d\xi d\eta = 2\hat{A}_\beta \sum_{k=1}^{\infty} \left[ \frac{\Gamma(\frac{n}{2} - \beta)\Gamma(\frac{n}{2} + \beta + k)}{\Gamma(\frac{n}{2} + \beta)\Gamma(\frac{n}{2} - \beta + k)} - 1 \right] \int |Y_k|^2 d\xi \quad (24)$$

$$2\hat{A}_\beta = 2^{-n-2\beta} \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2} + 1)} \frac{\Gamma(1-\beta)}{\beta \Gamma(\frac{n}{2} - \beta)}$$

*Proof.* Observe that by using the calculations for the Hardy-Littlewood-Sobolev inequality on the sphere (see [4], page 307) when  $0 < \lambda < n$

$$\int_{S^n \times S^n} \frac{|F(\xi) - F(\eta)|^2}{|\xi - \eta|^\lambda} d\xi d\eta = 2A_\lambda \sum_{k=1}^{\infty} \left[ 1 - \frac{\Gamma(n - \frac{\lambda}{2})\Gamma(\frac{\lambda}{2} + k)}{\Gamma(\frac{\lambda}{2})\Gamma(n - \frac{\lambda}{2} + k)} \right] \int |Y_k|^2 d\xi$$

$$A_\lambda = \int |\xi - \eta|^{-\lambda} d\xi d\eta = 2^{-\lambda} \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})}$$

Since the integral is well-defined for  $0 < \lambda < n + 2$ , analytic continuation gives the desired result for the Lemma.  $\square$

Since this smoothing form precisely captures the Hardy-Littlewood-Sobolev coefficients for expansion in spherical harmonics, a new representation can be given for the Hardy-Littlewood-Sobolev inequality on the sphere.

**Theorem 9.** For  $1 < p < 2$ ,  $\lambda = 2n/p'$ ,  $0 < \beta < \min\{1, n/2\}$  and  $q = 2n/(n - 2\beta)$  with  $d\xi =$  normalized surface measure

$$\int_{S^n} |F|^2 d\xi - \left( \int_{S^n} |F|^p d\xi \right)^{2/p} \leq \frac{1}{2A_\lambda} \int_{S^n \times S^n} \frac{|F(\xi) - F(\eta)|^2}{|\xi - \eta|^\lambda} d\xi d\eta \quad (25)$$

$$\left( \int_{S^n} |F|^q d\xi \right)^{2/q} - \int_{S^n} |F|^2 d\xi \leq \frac{1}{2\hat{A}_\beta} \int_{S^n \times S^n} \frac{|F(\xi) - F(\eta)|^2}{|\xi - \eta|^{n+2\beta}} d\xi d\eta \quad (26)$$

Further setting  $\int |F|^2 d\xi = 1$

$$\int_{S^n} |F|^2 \ln |F| d\xi \leq \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n+1}{2})} \int_{S^n \times S^n} \frac{|F(\xi) - F(\eta)|^2}{|\xi - \eta|^n} d\xi d\eta \quad (27)$$

$$= \sum_{k=1}^{\infty} \Delta_{k,n} \int_{S^n} |Y_k|^2 d\xi$$

$$\Delta_{k,n} = \frac{n}{2} \sum_{m=0}^{k-1} \frac{1}{m + \frac{n}{2}}$$

*Proof.* These estimates follow clearly from the Hardy-Littlewood-Sobolev inequality and the Aronszajn-Smith representation given above (see the section on “sharp Sobolev inequalities” in [1] and the discussion related to Theorem 1 in [4]).

$$\left( \int_{S^n} |F(\xi)|^q d\xi \right)^{2/q} \leq \sum_{k=0}^{\infty} \frac{\Gamma(\frac{n}{q})\Gamma(\frac{n}{q'} + k)}{\Gamma(\frac{n}{q'})\Gamma(\frac{n}{q} + k)} \int_{S^n} |Y_k|^2 d\xi \quad (28)$$

$\square$

## 5. DENSITY FUNCTIONAL THEORY AND PITT'S INEQUALITY

In density functional theory an object of interest is the Coulomb interaction energy

$$E_c(\psi) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{1}{|x - y|} |\psi(x, y)|^2 dx dy \quad (29)$$

Observe that the structure of Pitt's inequality will provide a sharp estimate for an upper bound for  $E_c(\psi)$  in terms of fractional Sobolev embedding on  $\mathbb{R}^{2n}$ . This analysis illustrates the principle that in general product functions may not provide an optimal estimation strategy.

**Theorem 10.** For  $f \in \mathcal{S}(\mathbb{R}^{2n})$ ,  $0 < \lambda < n$ ,  $\lambda = 2\alpha$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy \leq C_\lambda \int_{\mathbb{R}^{2n}} |(-\Delta/4\pi^2)^{\alpha/2} f|^2 dx dy \quad (30)$$

$$C_\lambda = (\pi/\sqrt{2})^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

This constant is sharp but not attained.

*Proof.* Inequality (30) is equivalent to the fractional integral inequality:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} \left| \int_{\mathbb{R}^n \times \mathbb{R}^n} [|x - u|^2 + |y - v|^2]^{-(2n-\alpha)/2} f(u, v) du dv \right|^2 dx dy$$

$$\leq C_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x, y)|^2 dx dy$$

$$C_\lambda = \pi^{-2n+2\alpha} \left[ \frac{\Gamma(\frac{2n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \right]^2 C_\lambda$$

By duality this is equivalent to

$$\int_{\mathbb{R}^{2n}} \left| \int_{\mathbb{R}^{2n}} |w - z|^{-(2n-\alpha)} |x - y|^{-\lambda/2} h(z) dz \right|^2 dw \leq C_\lambda \int_{\mathbb{R}^{2n}} |h(z)|^2 dz$$

where  $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$  and  $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ . By integrating out the free variable on the left-hand side, the inequality becomes

$$\int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} h(z) |x - y|^{-\lambda/2} |z - w|^{-(2n-2\alpha)} |u - v|^{-\lambda/2} h(w) dz, dw$$

$$\leq H_\lambda \int_{\mathbb{R}^{2n}} |h(z)|^2 dz$$

$$C_\lambda = \pi^{-n+2\alpha} \Gamma(n - \alpha) / \Gamma(\alpha) H_\lambda$$

To analyze the left-hand side, consider the rotation

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} \mathbb{1}_n & -\frac{1}{\sqrt{2}} \mathbb{1}_n \\ \frac{1}{\sqrt{2}} \mathbb{1}_n & -\frac{1}{\sqrt{2}} \mathbb{1}_n \end{bmatrix}$$

where  $\mathbb{1}_n$  is the identity matrix on  $\mathbb{R}^n$  and let  $P$  denote the projection on the first  $n$  variables. Then the left-hand side corresponds to

$$(\sqrt{2})^{-\lambda} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} h(z) |PRz|^{-\lambda/2} |z - w|^{-(2n-2\alpha)} |PRw|^{-\lambda/2} h(w) dz dw ;$$

by changing variables,  $z \rightarrow Rz$  and relabeling with the observation on matrices

$$R^{-2} = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}$$

this term can be rewritten as

$$(\sqrt{2})^{-\lambda} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} h(y_1 - x) |x|^{-\lambda/2} |z - w|^{-(2n-2\alpha)} |u|^{-\lambda/2} h(v_1 - u) dz dw$$

Applying Young's inequality in the variables  $y$  and  $v$  provides the upper bound

$$K_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} \bar{h}(x) |x|^{-\lambda/2} |x - u|^{-\lambda/2} \bar{h}(u) du dv$$

where

$$\bar{h}(x) = \left[ \int_{\mathbb{R}^n} |h(y, -x)|^2 dy \right]^{1/2}$$

$$K_\lambda = (\sqrt{2})^{-\lambda} \int_{\mathbb{R}^n} (1 + |y|^2)^{-(n-\lambda/2)} dy = 2^{-\lambda/2} \pi^{n/2} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})}$$

Pitt's inequality completes the argument:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \bar{h}(x) |x|^{-\lambda/2} |x - u|^{-(n-\lambda)} |u|^{-\lambda/2} \bar{h}(u) du dv \leq E_\lambda \int_{\mathbb{R}^n} |\bar{h}(x)|^2 dx$$

$$E_\lambda = \pi^{n/2} \frac{\Gamma(\frac{\lambda}{2})}{\Gamma(\frac{n-\lambda}{2})} \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

and

$$\int_{\mathbb{R}^n} |\bar{h}(x)|^2 dx = \int_{\mathbb{R}^{2n}} |h(x, y)|^2 dx dy$$

Tracing through all the steps in calculating the optimal constant gives:

$$\begin{aligned} C_\lambda &= \pi^{-n+\lambda} \frac{\Gamma(n - \frac{\lambda}{2})}{\Gamma(\frac{\lambda}{2})} H_\lambda \\ &= \pi^{-n+\lambda} \frac{\Gamma(n - \frac{\lambda}{2})}{\Gamma(\frac{\lambda}{2})} K_\lambda E_\lambda \\ &= \pi^{-n+\lambda} 2^{-\lambda/2} \frac{\Gamma(n - \frac{\lambda}{2})}{\Gamma(\frac{\lambda}{2})} \left[ \pi^{n/2} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})} \right] \left[ \pi^{n/2} \frac{\Gamma(\frac{\lambda}{2})}{\Gamma(\frac{n-\lambda}{2})} \right] \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2 \\ &= (\pi^2/2)^{\lambda/2} \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2 \end{aligned}$$

□

The calculation above expresses a mixing of radial symmetry and product structure which can be outlined in the following lemma.

**Lemma.** For  $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ ,  $0 < \lambda < n$ ,  $\lambda = 2\alpha$

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} |x|^{-\lambda} |f(x, y)|^2 dx dy \leq D_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^m} |(-\Delta/4\pi^2)^{\alpha/2} f|^2 dx dy \quad (31)$$

$$D_\lambda = \pi^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

*Proof.* Inequality (31) is equivalent to the fractional integral inequality:

$$\begin{aligned} & \int_{\mathbb{R}^n \times \mathbb{R}^m} |x|^{-\lambda} \left| \int_{\mathbb{R}^n \times \mathbb{R}^m} [|x-u|^2 + |v|^2]^{-(n+m-3\alpha)/2} f(u, v) du dv \right|^2 dx dy \\ & \leq G_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^m} |f(x, y)|^2 dx dy \\ & D_\lambda = \pi^{-n-m+2\alpha} \left[ \frac{\Gamma(\frac{n+m-\alpha}{2})}{\Gamma(\frac{\alpha}{2})} \right]^2 G_\lambda \end{aligned}$$

By duality this inequality is equivalent to

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \left| \int_{\mathbb{R}^n \times \mathbb{R}^m} |w-z|^{-(n+m-\alpha)} |x|^{-\lambda/2} h(z) dz \right|^2 dw \leq G_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^m} |h|^2 dz$$

where  $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$  and  $w = (u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ . After integrating out the free variable on the left-hand side, the inequality becomes

$$\begin{aligned} & \int_{\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}} h(z) |x|^{-\lambda/2} |z-w|^{-(n+m-2\alpha)} |u|^{-\lambda/2} h(w) dz dw \\ & \leq H_\lambda \int_{\mathbb{R}^{n+m}} |h(z)|^2 dz \\ & D_\lambda = \pi^{-(n+m)/2+2\alpha} \Gamma\left(\frac{n+m}{2} - \alpha\right) / \Gamma(\alpha) H_\lambda \end{aligned}$$

Applying Young's inequality in the variables  $y$  and  $v$  provides the upper bound for the left-hand side

$$K_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} \bar{h}(x) |x|^{-\lambda/2} |x-u|^{-(n-\lambda)} |u|^{-\lambda/2} \bar{h}(u) dx du$$

where

$$\begin{aligned} \bar{h}(x) &= \left[ \int_{\mathbb{R}^m} |h(x, y)|^2 dy \right]^{1/2} \\ K_\lambda &= \int_{\mathbb{R}^m} (1 + |y|^2)^{-(n+m-\lambda)/2} dy = \pi^{m/2} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(\frac{n+m-\lambda}{2})} \\ & \int_{\mathbb{R}^n \times \mathbb{R}^n} \bar{h}(x) |x|^{-\lambda/2} |x-u|^{-(n-\lambda)} |u|^{-\lambda/2} \bar{h}(u) dx du \\ & \leq \pi^{n/2} \frac{\Gamma(\frac{\lambda}{2})}{\Gamma(\frac{n-\lambda}{2})} \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2 \int_{\mathbb{R}^n} |\bar{h}(x)|^2 dx \end{aligned}$$

Tracing through all the steps results in

$$D_\lambda = \pi^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

which is independent of the dimension  $m$ . □

The lemma allows a more direct proof of Theorem 10 but the initial argument provides better understanding of the technical structure of the proof. Moreover that structure suggests that control of forms such as the Coulomb interaction energy by fractional smoothing is more an  $\mathbb{R}^n$  result than an  $\mathbb{R}^{2n}$  result as given in Theorem 10. This characterization is made explicit by combining Pitt's inequality with the Aronszajn-Smith formula.

**Theorem 11.** For  $f \in \mathcal{S}(\mathbb{R}^{2n})$ ,  $0 < \lambda < n$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy \leq F_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi - \eta|^\lambda |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \quad (32)$$

For  $0 < \lambda < \min(2, n)$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy \leq G_\lambda \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} |x - y - u - v|^{-n-\lambda} |f(x, y) - f(u, v)|^2 dx dy du dv \quad (33)$$

$$F_\lambda = (\pi/2)^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2; \quad G_\lambda = (\pi/2)^{-n/2} \left( \frac{\lambda}{4} \right) \frac{\Gamma(\frac{n+\lambda}{2})}{\Gamma(1 - \frac{\lambda}{2})} \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

*Proof.* Using the rotation  $R$  from the proof of Theorem 10 and Pitt's inequality

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy &= 2^{-\lambda/2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |PRz|^{-\lambda} |f(x, y)|^2 dx dy \\ &= 2^{-\lambda/2} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x|^{-\lambda} |g(x, y)|^2 dx dy \leq 2^{-\lambda/2} D_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi|^\lambda |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &= 2^{-\lambda} D_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi - \eta|^\lambda |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \end{aligned}$$

where  $z = (x, y)$ ,  $g(x, y) = f(R^{-1}z)$  and for  $w = (\xi, \eta)$

$$\begin{aligned} \widehat{g}(\xi, \eta) &= \int_{\mathbb{R}^{2n}} e^{2\pi i w z} g(z) dz = \int_{\mathbb{R}^{2n}} e^{2\pi i w z} f(R^{-1}z) dz \\ &= \int_{\mathbb{R}^{2n}} e^{2\pi i (R^{-1}w) z} f(z) dz = \widehat{f}(R^{-1}w). \end{aligned}$$

Here

$$D_\lambda = \pi^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2 \implies F_\lambda = (\pi/2)^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2.$$

Using the Aronszajn-Smith formula

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi|^\lambda |\widehat{g}(\xi, \eta)|^2 d\xi d\eta &= \frac{1}{E_\lambda} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \frac{|g(x, y) - g(u, v)|^2}{|x - u|^{n+\lambda}} dx dy du dv \\ &= \frac{2^{(n+\lambda)/2}}{E_\lambda} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \frac{|f(x, y) - f(u, v)|^2}{|x - y - u + v|^{n+\lambda}} dx dy du dv \end{aligned}$$

with

$$E_\lambda = \frac{4}{\lambda} \pi^{\frac{n}{2} + \lambda} \frac{\Gamma(1 - \frac{\lambda}{2})}{\Gamma(\frac{n+\lambda}{2})}.$$



Tracing back on the varied constants

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy &\leq 2^{-\lambda/2} D_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi|^\lambda |\widehat{g}(\xi, \eta)|^2 d\xi d\eta \\ &= G_\lambda \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \frac{|f(x, y) - f(u, v)|^2}{|x - y - u + v|^{n+\lambda}} dx dy du dv \end{aligned}$$

with

$$G_\lambda = (\pi/2)^{-n/2} \left( \frac{\lambda}{4} \right) \frac{\Gamma(\frac{n+\lambda}{2})}{\Gamma(1 - \frac{\lambda}{2})} \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

□

**Corollary** (logarithmic uncertainty). *For  $f \in \mathcal{S}(\mathbb{R}^n)$*

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \ln |x - y| |f(x, y)|^2 dx dy + \int_{\mathbb{R}^n \times \mathbb{R}^n} \ln |\xi - \eta| |\widehat{f}(\xi, \eta)|^2 d\xi d\eta \\ \geq D \int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x, y)|^2 dx dy \end{aligned} \quad (34)$$

$$D = \psi(n/4) - \ln(\pi/2), \quad \psi = (\ln \Gamma)'$$

*Proof.* Observe that inequality (32) is an equality at  $\lambda = 0$  so differentiate the inequality at  $\lambda = 0$ ; one can also derive this logarithmic weighted form directly from the original logarithmic uncertainty form using the rotation  $R$  above (see section 2 in [6]). □

More generally, such inequalities as above extend to multidimensional components. For  $A \in \mathbb{R}^n$  with  $|A| = 1$  and  $x_k \in \mathbb{R}^n$

$$\begin{aligned} \int_{\mathbb{R}^{mn}} \ln \left| \sum A_k x_k \right| |f(x_1, \dots, x_m)|^2 dx + \int_{\mathbb{R}^{mn}} \ln \left| \sum A_k \xi_k \right| |\widehat{f}(\xi_1, \dots, \xi_m)|^2 d\xi \\ \geq [\psi(n/4) - \ln \pi] \int_{\mathbb{R}^{mn}} |f(x_1, \dots, x_m)|^2 dx \end{aligned} \quad (35)$$

Note that as in the Lemma following Theorem 10 the constant depends only on the component dimension  $n$ .

As one might expect by association with the Coulomb interaction energy, the functional

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy$$

has an underlying conformal invariance: let  $f(x, y) = (1 + |x|^2)^{-n/p} (1 + |y|^2)^{-n/p} F(\xi, \eta)$  for  $(\xi, \eta) \in S^n \times S^n$  and  $\frac{\lambda}{2} + \frac{2n}{p} = n$  (e.g.,  $p = 2n/[n - (\lambda/2)] > 2$ ); then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |f(x, y)|^2 dx dy = C_{\lambda, n} \int_{S^n \times S^n} |\xi - \eta|^{-\lambda} |F(\xi, \eta)|^2 d\xi d\eta \quad (36)$$

$$C_{\lambda, n} = 2^\lambda \pi^n \left[ \Gamma(n/2) / \Gamma(n) \right]^2$$

where  $d\xi, d\eta$  denote normalized surface measure with the map from  $\mathbb{R}^n$  to  $S^n$  defined by

$$\xi = \left( \frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right), \quad d\xi = \pi^{-n/2} \left[ \Gamma(n)/\Gamma(n/2) \right] (1+|x|^2)^{-n} dx$$

$$|x-y| = \frac{1}{2} |\xi - \eta| \left[ (1+|x|^2)(1+|y|^2) \right]^{1/2}$$

This invariance is highly suggestive to examine the case of product states where one expects that optimal constants will be attained in contrast to the earlier results obtained by relating the functional to Pitt's inequality. Then for  $f(x, y) = \varphi(x)\varphi(y)$

**Theorem 12.** For  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  and  $p = 2n/[n - (\lambda/2)]$ ,  $0 < \lambda < n$

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \varphi^2(x) |x-y|^{-\lambda} \varphi^2(y) dx dy &\leq b_{\lambda,n} \left[ \int_{\mathbb{R}^n} |\varphi|^p dx \right]^{4/p} \\ &\leq b_{\lambda,n} d_{\lambda,n} \left[ \int_{\mathbb{R}^n} |(-\Delta/4\pi^2)^{\lambda/8} \varphi|^2 dx \right]^2 \\ b_{\lambda,n} &= \pi^{n(-\frac{2}{p})} \frac{\Gamma(\frac{2n}{p} - \frac{n}{2})}{\Gamma(\frac{2n}{p})} \left[ \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right]^{\frac{2}{p}-1} \\ d_{\lambda,n} &= \left[ \pi^{n/p-n/2} \left[ \frac{\Gamma(n/p)}{\Gamma(n/p')} \right] \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{1-2/p} \right]^2 \\ b_{\lambda,n} d_{\lambda,n} &= \frac{\Gamma(\frac{2n}{p} - \frac{n}{2})}{\Gamma(\frac{2n}{p})} \left[ \frac{\Gamma(n/p)}{\Gamma(n/p')} \right]^2 \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{1-2/p} \end{aligned} \tag{37}$$

*Proof.* This result follows from successive applications of the Hardy-Littlewood-Sobolev inequality. For the first step, observe that if  $q = n/[n - (\lambda/2)]$ , then  $\lambda = 2n/q'$  which results in the first part of inequality (37); the second step is an equivalent form taken from the Lemma in section 6 below.  $\square$

As a consequence of inverting the fractional smoothing in this result, one can find an equivalent representation in terms of a multilinear Hardy-Littlewood-Sobolev inequality.

**Theorem 13.** For  $\varphi \in L^2(\mathbb{R}^n)$  and  $0 < \lambda < n$

$$\begin{aligned} \int_{\mathbb{R}^{6n}} \varphi(u_1) \varphi(u_2) \varphi(v_1) \varphi(v_2) \prod_k |x - u_k|^{-(n-\lambda/4)} \prod_k (|y - v_k|)^{-(n-\lambda/4)} |x-y|^{-\lambda} dx dy du dv \\ \leq A_\lambda \left( \int_{\mathbb{R}^n} |\varphi|^2 dx \right)^2 \end{aligned} \tag{38}$$

$$A_\lambda = \pi^{-4n/p} \frac{\Gamma[n(\frac{2}{p} - \frac{1}{2})]}{\Gamma[\frac{2n}{p}]} \left[ \frac{\Gamma[n(1 + \frac{2}{p})/4]}{\Gamma[n(1 - \frac{2}{p})/4]} \right]^4 \left[ \frac{\Gamma(n/p)}{\Gamma(n/p')} \right]^2 \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{1-2/p}$$

The most striking feature of this inequality is that it comes from two successive applications of sharp conformally invariant inequalities but results in a form that is not clearly amenable to application of symmetry methods to determine extremal functions. This obstruction is due

to the interior integrals over the  $(x, y)$  variables. In one case,  $\lambda = 4$  for  $n > 4$ , an extremal function for inequality (36) is given by

$$c(1 + |x|^2)^{-(n/2-1)}$$

which then allows the extremal function for inequality (37) to be obtained as the solution for the convolution equation

$$\varphi * |x|^{-(n-1)} = c(1 + |x|^2)^{-(n/2-1)}$$

Determining the physical behavior and mathematical description for many body dynamics is generally hard — because both the complexity of symmetry and the possible combination of interaction increase substantially. A simple example that results from an application of the Hardy-Littlewood-Sobolev inequality and could relate to multiparticle interaction is given by

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} |x + y + z|^{-\lambda} f(x) f(y) f(z) dx dy dz \right| \leq B_\lambda [\|f\|_{L^p(\mathbb{R}^n)}]^3 \quad (39)$$

for  $\lambda = 3n/p'$  and  $p' > 3$ . But how the optimal constant  $B_\lambda$  could be calculated is unclear. The critical question to understand here is the character of metrics that span multiple points.

By adapting Theorem 11 to the case of product functions, a novel representation of Coulomb interaction forms is outlined which appears to be formulated using the structure of the Hardy-Littlewood-Sobolev inequality but is in fact a realization of Pitt's inequality. While the most direct proof of this results is obtained from Pitt's inequality, an alternative proof can be given using a combination of the Hardy-Littlewood-Sobolev inequality, the reverse Hardy-Littlewood-Sobolev inequality and the Hausdorff-Young inequality. This appears to be one of the first examples where the reverse Hardy-Littlewood-Sobolev inequality has an interesting application. Part of this inequality was already used in Carneiro's thesis (see pages 3133-3134 in [15]).

**Theorem 14.** For  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ ,  $0 < \lambda < n$

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |\varphi(x)|^2 |\varphi(y)|^2 dx dy \leq F_\lambda \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi - \eta|^\lambda |\widehat{\varphi}(\xi)|^2 |\widehat{\varphi}(\eta)|^2 d\xi d\eta \quad (40)$$

*Alternative proof (without sharp constants).* Use the first line of inequality (37) to obtain

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^{-\lambda} |\varphi(x)|^2 |\varphi(y)|^2 dx dy \leq b_{\lambda,n} \left[ \int_{\mathbb{R}^n} |\varphi|^p dx \right]^{4/p}$$

for  $p = 2n/(n - \lambda/2) > 2$  using the Hardy-Littlewood-Sobolev inequality; now apply the Hausdorff-Young inequality to obtain

$$\leq b_{\lambda,n} c_{p,n} \left[ \int_{\mathbb{R}^n} |\widehat{\varphi}|^{p'} \right]^{4/p'}, \quad p' = 2n/\left(n + \frac{\lambda}{2}\right)$$

and now apply the reverse Hardy-Littlewood-Sobolev inequality to find

$$\leq b_{\lambda,n} c_{p,n} e_{\lambda,n} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\xi - \eta|^\lambda |\widehat{\varphi}(\xi)|^2 |\widehat{\varphi}(\eta)|^2 d\xi d\eta$$

for  $0 < \lambda < n$  and  $p = 2n/(n - \lambda/2)$ . □

**Corollary.** For  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ ,  $n > 2$  and  $\Omega(\xi) = \sum_{i < j} |\xi_i - \xi_j|^2$

$$\int_{\mathbb{R}^{mn}} \sum_{i < j} |x_i - x_j|^{-2} \prod_{k=1}^m |\varphi(x_k)|^2 dx \leq \frac{4\pi^2}{(n-2)^2} \int_{\mathbb{R}^{mn}} \Omega(\xi) \prod_{k=1}^m |\widehat{\varphi}(\xi_k)|^2 d\xi \quad (41)$$

The challenge of extending the Hardy-Littlewood-Sobolev inequality both in terms of multiple interaction and retaining “reverse estimates” suggests the following inequalities that extend equation (39) for the case  $\lambda = mn/p'$ ,  $p' > m$  and adapt similar arguments used for the proof of Theorem 11.

**Theorem 15.** For  $f \in \mathcal{S}(\mathbb{R}^{mn})$ ,  $0 < \lambda < n$

$$\int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} |f(x_1, \dots, x_m)|^2 dx \leq F_\lambda \int_{\mathbb{R}^{mn}} |\sum \xi_k|^\lambda |\widehat{f}(\xi_1, \dots, \xi_m)|^2 d\xi \quad (42)$$

$$\int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} \prod |\varphi(x_k)|^2 dx \leq F_\lambda \int_{\mathbb{R}^{mn}} |\sum \xi_k|^\lambda \prod |\widehat{\varphi}(\xi_k)|^2 d\xi \quad (43)$$

$$F_\lambda = (\pi/m)^\lambda \left[ \frac{\Gamma(\frac{n-\lambda}{4})}{\Gamma(\frac{n+\lambda}{4})} \right]^2$$

For  $\lambda = mn/p'$ ,  $p' > m$  ( $p = \frac{mn}{mn-\lambda}$ ,  $q = \frac{mn}{mn+\lambda}$ )

$$\int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} \prod |f(x_k)| dx \leq c_1 \left[ \|f\|_{L^p(\mathbb{R}^n)} \right]^m \quad (44)$$

$$\int_{\mathbb{R}^{mn}} |\sum \xi_k|^\lambda \prod |g(\xi_k)| d\xi \geq c_2 \left[ \|g\|_{L^q(\mathbb{R}^n)} \right]^m \quad (45)$$

*Proof.* The proof of inequality (42) follows the argument used in the proof of Theorem 11 and inequality (32). An alternate proof of (43) follows the method of Theorem 14 using the multilinear Hardy-Littlewood-Sobolev inequalities (44) and (45). The first inequality is obtained by iterating the following reduction so that the estimate depends on the case  $m = 2$  which is the original Hardy-Littlewood-Sobolev inequality. First, use rearrangement and symmetrization to reduce the problem to the case where  $f$  is radial decreasing:

$$\int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} \prod |f(x_k)| dx \leq \int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} \prod f^*(x_k) dx$$

where  $f^*$  is the equimeasurable radial decreasing rearrangement of  $|f|$  on  $\mathbb{R}^n$ . Then observe that

$$f^*(x) \leq c \|f\|_{L^p(\mathbb{R}^n)} |x|^{-n/p}, \quad |x|^{-\lambda} * |x|^{-n/p} = c |x|^{-\lambda+n/p'}$$

so that

$$\begin{aligned} & \int_{\mathbb{R}^{mn}} |\sum x_k|^{-mn/p'} \prod f^*(x_k) dx_1 \dots dx_m \\ & \leq c \|f\|_{L^p(\mathbb{R}^n)} \int_{\mathbb{R}^{(m-1)n}} |\sum x_k|^{-(m-1)n/p'} \prod f^*(x_k) dx_1 \dots dx_{m-1} \end{aligned}$$

Continuing this iteration, one obtains the reduction

$$\int_{\mathbb{R}^{mn}} |\sum x_k|^{-mn/p'} \prod f^*(x_k) dx \leq c \left[ \|f\|_{L^p(\mathbb{R}^n)} \right]^{m-2} \int_{\mathbb{R}^n \times \mathbb{R}^n} f^*(x) |x-y|^{-2n/p'} f^*(y) dx dy$$

and the proof of inequality (44) is obtained by using the Hardy-Littlewood-Sobolev inequality. The second inequality (45) is obtained by iterating a similar reduction to the one just used so that the estimate depends on the case  $m = 2$  which is the reverse Hardy-Littlewood-Sobolev

inequality (see appendix). Again use rearrangement and symmetrization to reduce the problem to the case where  $g$  is radial decreasing:

$$\int_{\mathbb{R}^{mn}} \left| \sum \xi_k \right|^\lambda \prod |g(\xi_k)| d\xi \geq \int_{\mathbb{R}^{mn}} \left| \sum \xi_k \right|^\lambda \prod g^*(\xi_k) d\xi$$

where  $g^*$  is the equimeasurable radial decreasing rearrangement of  $|g|$  on  $\mathbb{R}^n$ . Here one uses the following variation on the Brascamp-Lieb-Luttinger rearrangement inequality for the function  $h$  being radial and increasing:

$$\begin{aligned} & \int_{\mathbb{R}^{mn}} h\left(\sum b_k x_k\right) \prod_{\ell=1}^N \left| g_\ell\left(\sum_k a_{\ell k} x_k\right) \right| dx_1 \dots dx_m \\ & \geq \int_{\mathbb{R}^{mn}} h\left(\sum b_k x_k\right) \prod_{\ell=1}^N g_\ell^*\left(\sum_k a_{\ell k} x_k\right) dx_1 \dots dx_m \end{aligned}$$

Now observe that

$$\begin{aligned} |g^*(x)|^{-\alpha} & \geq c \left[ \|g\|_{L^q(\mathbb{R}^n)} \right]^{-\alpha} |x|^{\alpha n/q} \\ \int_{\mathbb{R}^n} g^*(x) |x-y|^\lambda dx & \geq c \int_{\mathbb{R}^n} [g^*(x)]^{1+\alpha} |x|^{\alpha n/q} |x-y|^\lambda dx \left[ \|g\|_{L^q(\mathbb{R}^n)} \right]^{-\alpha} \\ & \geq c \| |x|^{\alpha n/q} |x-y|^\lambda \|_{L^r(\mathbb{R}^n)} \| (g^*)^{1+\alpha} \|_{L^{r'}(\mathbb{R}^n)} \left[ \|g\|_{L^q(\mathbb{R}^n)} \right]^{-\alpha}; \end{aligned}$$

set  $r = -s < 0$ ,  $r' = s/(s+1)$  with  $(1+\alpha)s/(1+s) = q$ . Then  $\|(g^*)^{1+\alpha}\|_{r'} = (\|g\|_q)^{1+\alpha}$ ; rewriting the relation for  $q$  gives  $s(1-q) + \alpha s = q$  which implies  $\alpha s < q$  since  $q < 1$ . Using the relation for  $q$  with respect to  $\lambda$ ,  $q = mn/(mn+\lambda)$ , three equivalent defining relations for  $\alpha$  and  $s$  can be given in terms of the input value for  $\lambda$ ;

$$s(1-q) + \alpha s = q; \quad \frac{1}{s} = (1+\alpha) \frac{\lambda}{mn} + \alpha; \quad \lambda + \alpha \frac{n}{q} - \frac{n}{s} = (m-1)\lambda/m$$

Any values of  $\alpha$  and  $s$  can be used in the following calculation as long as  $\alpha s < q$  and  $s\lambda < n$ ; the first condition holds in general, and the second will hold for  $\alpha \geq 1$  since then  $s < 1$ , and already  $\lambda < n$ . Then

$$\begin{aligned} \| |x|^{\alpha n/q} |x-y|^\lambda \|_{L^r(\mathbb{R}^n)} & \geq \left[ \int_{\mathbb{R}^n} |x|^{-s\alpha n/q} |x-y|^{-s\lambda} dx \right]^{-1/s} \\ & = c \left[ |y|^{-(s\alpha n/q + s\lambda - n)} \right]^{-1/s} = c |y|^{\alpha n/q + \lambda - n/s} \\ & = c |y|^{(m-1)\lambda/m} = c |y|^{(m-1)n/p'}. \end{aligned}$$

Now

$$\begin{aligned} & \int_{\mathbb{R}^{mn}} \left| \sum x_k \right|^{mn/p'} \prod g^*(x_k) dx_1 \dots dx_m \\ & \geq c \|g\|_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^{(m-1)n}} \left| \sum x_k \right|^{-(m-1)n/p'} \prod g^*(x_k) dx_1 \dots dx_{m-1} \end{aligned}$$

Continuing this iteration, one obtains the reduction

$$\begin{aligned} & \int_{\mathbb{R}^{mn}} |\sum x_k|^{mn/p'} \prod g^*(x_k) dx \\ & \geq c \left[ \|g\|_{L^q(\mathbb{R}^n)} \right]^{m-2} \int_{\mathbb{R}^n \times \mathbb{R}^n} g^*(x) |x+y|^{2n/p'} g^*(y) dx dy \end{aligned}$$

and the proof of inequality (45) is obtained from the reverse Hardy-Littlewood-Sobolev inequality.

These two expanded Hardy-Littlewood-Sobolev estimates combined with the Hausdorff-Young inequality give a proof without sharp constants for inequality (43). Choose  $p$  so that for  $0 < \lambda < n$ ,  $\lambda = mn/p'$ . Then using (44), (45) and the Hausdorff-Young inequality for  $r = 2mn/(mn - \lambda) = 2p > 2$  and  $r' = 2mn/(mn + \lambda) = 2q < 2$

$$\begin{aligned} & \int_{\mathbb{R}^{mn}} |\sum x_k|^{-\lambda} \prod |\varphi(x_k)|^2 dx \leq c \left[ \|\varphi\|_{L^r(\mathbb{R}^n)} \right]^{2m} \\ & \leq c \left[ \|\widehat{\varphi}\|_{L^{r'}(\mathbb{R}^n)} \right]^{2m} \leq c \int_{\mathbb{R}^{mn}} |\sum \xi_k|^\lambda \prod |\widehat{\varphi}(\xi_k)|^2 d\xi . \end{aligned}$$

Here  $c$  is a generic constant, and the proof of (43) is complete.  $\square$

## 6. HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

A natural question that underlines the development described here and in recent papers — *identify the intrinsic character of the Hardy-Littlewood-Sobolev inequality*. The starting point would be the fractional integral defined by the Riesz potential

$$f \in L^p(\mathbb{R}^n) \rightsquigarrow \frac{1}{|x|^\lambda} * f \in L^{p'}(\mathbb{R}^n) \quad (46)$$

with  $1 < p < 2$ ,  $1/p + 1/p' = 1$  and  $\lambda = 2n/p'$ . Here conformal invariance enables calculation of the sharp constant for the operator norm [19]:

$$\begin{aligned} & \left\| \frac{1}{|x|^\lambda} * f \right\|_{L^{p'}(\mathbb{R}^n)} \leq A_p \|f\|_{L^p(\mathbb{R}^n)} \quad (47) \\ & A_p = \pi^{n/p'} \frac{\Gamma[n(\frac{1}{p} - \frac{1}{2})]}{\Gamma(n/p)} \left[ \frac{\Gamma(n)}{\Gamma(n/2)} \right]^{\frac{2}{p}-1} \end{aligned}$$

Later it was recognized that an inherent axial symmetry would lead to an equivalent representation on the Liouville-Beltrami model for hyperbolic space and provide a quick determination of the extremal functions for the optimal inequality. This calculation demonstrated how hyperbolic symmetry is embedded in the conformal structure of the Riesz functional ([3]).

Because of the inequality's structure as a map from a space to its dual, one can utilize the *square-integrable paradigm* to give an equivalent representation for the Hardy-Littlewood-Sobolev inequality in terms of fractional smoothness.

**Lemma.** For  $f \in \mathcal{S}(\mathbb{R}^n)$ ,  $1 < p < 2$ ,  $\alpha = n(1/p - 1/2)$  and  $1/p + 1/p' = 1$

$$\int_{\mathbb{R}^n} |f|^2 dx \leq C_p \left[ \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/2} f \right|^p dx \right]^{2/p} \quad (48)$$

$$\left[ \int_{\mathbb{R}^n} |g|^{p'} dx \right]^{2/p'} \leq C_p \int_{\mathbb{R}^n} \left| (-\Delta/4\pi^2)^{\alpha/2} f \right|^2 dx \quad (49)$$

$$C_p = \pi^{n/p' - n/2} \left[ \Gamma(n/p') / \Gamma(n/p) \right] \left[ \Gamma(n) / \Gamma(n/2) \right]^{2/p - 1}$$

And the Hardy-Littlewood-Sobolev inequality can be viewed as a positive-definite symmetric bilinear quadratic form:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) |x - y|^{-\lambda} f(y) dx dy \leq A_p (\|f\|_p)^2, \quad \lambda = 2n/p' \quad (50)$$

These inequalities suggest that the defining structure of the Hardy-Littlewood-Sobolev inequality should be equally identified with its representation in terms of fractional smoothness rather than simply in terms of the Riesz potential. To be more explicit, the Hardy-Littlewood-Sobolev inequality can be understood in terms of control determined by fractional smoothness while the role of the Riesz potential may be most useful in calculating formulas for sharp constants to characterize that control. This perspective provides critical insight for extending both the multilinear character and the domain manifold structure for which one can calculate sharp constants for the Hardy-Littlewood-Sobolev inequality.

## APPENDIX

### 1. Explicit calculation for an integral.

For  $f \in \mathcal{S}(\mathbb{R}^n)$  and  $0 < \alpha < 2$ , consider

$$\left[ (-\Delta/4\pi^2)^{\alpha/2} f \right](x) = \gamma_{\alpha,n} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy \quad (51)$$

$$\gamma_{\alpha,n} = (\alpha/2) \pi^{-\alpha-n/2} \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})} = 2/D_\alpha$$

To verify this constant, apply the Fourier transform to this equation

$$\begin{aligned}
|\xi|^\alpha \widehat{f}(\xi) &= \gamma_{\alpha,n} \mathcal{F} \left[ \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy \right] \\
&= \gamma_{\alpha,n} \mathcal{F} \left[ \int_{\mathbb{R}^n} \frac{f(x) - f(x+y)}{|y|^{n+\alpha}} dy \right] \\
&= \gamma_{\alpha,n} \int_{\mathbb{R}^n} \frac{[1 - e^{-2\pi i y \cdot \xi}]}{|y|^{n+\alpha}} dy \widehat{f}(\xi) \\
&= \gamma_{\alpha,n} \left[ \int_{\mathbb{R}^n} \frac{[1 - e^{-2\pi i y \cdot \hat{\eta}}]}{|y|^{n+\alpha}} dy \right] |\xi|^\alpha \widehat{f}(\xi), \quad |\eta| = 1 \\
(\gamma_{\alpha,n})^{-1} &= \int_{\mathbb{R}^n} \frac{[1 - e^{-2\pi i y \cdot \hat{\eta}}]}{|y|^{n+\alpha}} dy \\
&= \int_0^\infty \left[ \frac{2\pi^{n/2}}{\Gamma(n/2)} - 2\pi w^{(2-n)/2} J_{(n-2)/2}(2\pi w) \right] w^{-1-\alpha} dw \\
&= (2\pi)^{\alpha+n/2} \int_0^\infty \left[ \left( 2^{n/2-1} \Gamma(n/2) \right)^{-1} - w^{(2-n)/2} J_{(n-2)/2}(w) \right] w^{-1-\alpha} dw
\end{aligned}$$

Observe that for real  $w$  the series expansion for the Bessel function is given by

$$J_\nu(w) = (w/2)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{(w/2)^{2k}}{k! \Gamma(\nu + k + 1)}$$

so that the integral above can be calculated using “integration by parts”:

$$\begin{aligned}
&\int_0^\infty \left[ \left( 2^{n/2-1} \Gamma(n/2) \right)^{-1} - w^{(2-n)/2} J_{(n-2)/2}(w) \right] w^{-1-\alpha} dw \\
&= -\frac{1}{\alpha} \int_0^\infty \left[ \left( 2^{n/2-1} \Gamma(n/2) \right)^{-1} - w^{(2-n)/2} J_{(n-2)/2}(w) \right] d(w^{-\alpha}) \\
&= -\frac{1}{\alpha} \int_0^\infty w^{-\alpha} \frac{d}{dw} \left[ w^{(2-n)/2} J_{(n-2)/2}(w) \right] dw \\
&= \frac{1}{\alpha} \int_0^\infty w^{-\alpha+(2-n)/2} J_{n/2}(w) dw \\
&= \frac{1}{\alpha} 2^{1-\alpha-n/2} \frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}.
\end{aligned}$$

Hence

$$\gamma_{\alpha,n} = (\alpha/2) \pi^{-\alpha-n/2} \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})} = 2/D_\alpha$$

Evaluation of the integral for the Bessel function is taken from Erdelyi, *Higher Transcendental Functions* (see Gradshteyn & Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, 1965, page 684, formula 14).



At first glance the appearance of the Aronszajn-Smith constant is unexpected, but it follows directly from the formula for real-valued functions:

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} dx dy = 2 \int_{\mathbb{R}^n} f(x) \left[ \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+\alpha}} dy \right] dx \quad (52)$$

Alternative arguments can be given to calculate this integral using Gaussian subordination and Green's theorem: for  $\eta \in S^{n-1}$

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{1}{|w|^{n+\alpha}} (1 - e^{-2\pi i w \cdot \eta}) dw \\ &= \int_{\mathbb{R}^n} \frac{1}{|w|^{n+\alpha}} (1 - \cos 2\pi w \cdot \eta) dw \\ &= \frac{\pi^{\frac{n+\alpha}{2}}}{\Gamma(\frac{n+\alpha}{2})} \int_{\mathbb{R}^n} (1 - \cos 2\pi w \cdot \eta) \int_0^\infty t^{\frac{n+\alpha}{2}-1} e^{-\pi t w^2} dt \\ &= \frac{\pi^{\frac{n+\alpha}{2}}}{\Gamma(\frac{n+\alpha}{2})} \int_0^\infty t^{\frac{n+\alpha}{2}-1} \int_{\mathbb{R}^n} (1 - \cos 2\pi w \cdot \eta) e^{-\pi t w^2} dw \\ &= \frac{\pi^{\frac{n+\alpha}{2}}}{\Gamma(\frac{n+\alpha}{2})} \int_0^\infty t^{\frac{\alpha}{2}-1} (1 - e^{-\pi/t}) dt \\ &= \frac{\pi^{\frac{n+\alpha}{2}}}{\Gamma(\frac{n+\alpha}{2})} \int_0^\infty t^{-\frac{\alpha}{2}-1} (1 - e^{-t}) dt \\ &= \frac{2}{\alpha} \frac{\pi^{\frac{n}{2}+\alpha}}{\Gamma(\frac{n+\alpha}{2})} \int_0^\infty t^{-\alpha/2} e^{-t} dt = \frac{2}{\alpha} \pi^{\frac{n}{2}+\alpha} \frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}. \end{aligned}$$

The positivity of the integrands justify the exchange of orders of integration using Fubini's theorem. A third argument can be given using distribution theory and Green's theorem.

$$\begin{aligned} & \int_{\mathbb{R}^n} \frac{1}{|w|^{n+\alpha}} (1 - \cos 2\pi w \cdot \eta) dw \\ &= \frac{1}{2} \left[ \alpha \left( \frac{n+\alpha}{2} - 1 \right) \right]^{-1} \int_{\mathbb{R}^n} \Delta \left( \frac{1}{|w|^{n+\alpha-2}} \right) (1 - \cos 2\pi w \cdot \eta) dw \\ &= \frac{1}{2} \left[ \alpha \left( \frac{n+\alpha}{2} - 1 \right) \right]^{-1} \int_{\mathbb{R}^n} \frac{1}{|w|^{n+\alpha-2}} \Delta (1 - \cos 2\pi w \cdot \eta) dw \\ &= 2\pi^2 \left[ \alpha \left( \frac{n+\alpha}{2} - 1 \right) \right]^{-1} \int_{\mathbb{R}^n} \frac{1}{|w|^{n+\alpha-2}} \cos 2\pi w \cdot \eta dw \\ &= 2\pi^2 \left[ \alpha \left( \frac{n+\alpha}{2} - 1 \right) \right]^{-1} \mathcal{F} \left[ \frac{1}{|w|^{n+\alpha-2}} \right] (\eta) \\ &= \frac{2\pi^{\frac{n}{2}+\alpha}}{\alpha} \frac{\Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}. \end{aligned}$$

An independent derivation using Pizzetti's formula and analytic continuation can be found in Landkof [18] (see formula (1.1.6) on page 46).

## 2. Global embedding and boundary value estimates.

Symmetrization on the multiplicative group  $\mathbb{R}_+$  allows one to obtain a direct relation between the estimates (14) and (15), here taken on  $\mathbb{R}^{n+1}$ . Consider for  $\lambda = (n-1)/2$  and  $f = |x|^{-\lambda}g$

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx &= \int_{\mathbb{R}^{n+1}} |\nabla(|x|^{-\lambda}g)|^2 dx = \int_{\mathbb{R}^{n+1}} |x|^{-2\lambda} \left| \nabla g - \lambda |x|^{-1} \hat{\mathbf{i}}_r \right|^2 dx \\ &= \int_{\mathbb{R}^{n+1}} |x|^{-2\lambda+n} \left[ |\nabla g|^2 - 2\lambda |x|^{-1} g \frac{\partial g}{\partial r} + \lambda^2 |x|^{-2} |g|^2 \right] dr dv \end{aligned}$$

View  $g$  as a function of  $r$  and  $\xi \in S^n$  with  $\nabla_s$  denoting the spherical gradient,  $r = |x|$ ,  $\hat{\mathbf{i}}_r = x/|x|$ , and  $d\sigma$  being standard surface measure on the unit sphere. Observe that  $D = r \frac{\partial}{\partial r}$  is the invariant gradient on  $\mathbb{R}_+$ . Then the expression above can be rewritten as:

$$\begin{aligned} &\int_{\mathbb{R}_+ \times S^n} \left[ |Dg|^2 + |\nabla_s g|^2 + \lambda^2 |g|^2 \right] \frac{dr}{r} d\sigma \\ &\geq \int_{\mathbb{R}_+ \times S^n} \left[ |Dg^*|^2 + |\nabla_s g^*|^2 + \lambda^2 |g^*|^2 \right] \frac{dr}{r} d\sigma \\ &= 2 \int_{\{0 < r < 1\} \times S^n} \left[ |Dg^*|^2 + |\nabla_s g^*|^2 + \lambda^2 |g^*|^2 \right] \frac{dr}{r} d\sigma \end{aligned}$$

where  $g^*$  denotes for each  $\xi \in S^n$  the non-negative equimeasurable symmetric decreasing rearrangement of  $|g(r, \xi)|$  away from the “origin”  $r = 1$  and as a function of  $r$ . Now the expression above may be rephrased for  $f_{\#} = |x|^{-\lambda}g^*$  as

$$\begin{aligned} &2 \int_{|x| \leq 1} |\nabla f_{\#}|^2 dx + 4\lambda \int_{|x| \leq 1} g^* \frac{\partial g^*}{\partial r} dr d\sigma \\ &= 2 \int_{|x| \leq 1} \left| \nabla(|x|^{-\lambda}g^*) \right|^2 dx + 4\lambda \int_{|x| \leq 1} g^* \frac{\partial g^*}{\partial r} dr d\sigma \\ &= 2 \int_{|x| \leq 1} |\nabla f_{\#}|^2 dx + 2\lambda \int_{S^n} |g^*(\xi)|^2 d\sigma \end{aligned}$$

Then

$$\begin{aligned} b_n \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx &\geq 2b_n \int_{|x| \leq 1} |\nabla f_{\#}|^2 dx + 2\lambda b_n \int_{S^n} |g^*(\xi)|^2 d\sigma \\ 2\lambda b_n &= \frac{(n-1)\pi^{-(n+1)/2}}{4} \Gamma\left(\frac{n-1}{2}\right) = \left[ 2\pi^{(n+1)/2} / \Gamma\left(\frac{n+1}{2}\right) \right]^{-1} = 1/\sigma(S^n) \end{aligned}$$

and

$$\begin{aligned}
b_n \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx &\geq 2b_n \int_{|x| \leq 1} |\nabla f_{\#}|^2 dx + \int_{S^n} |g^*(\xi)|^2 d\xi \\
&\geq 2b_n \int_{|x| \leq 1} |\nabla u_{\#}|^2 dx + \int_{S^n} |g^*(\xi)|^2 d\xi \\
&\geq \left[ \int_{S^n} |g^*(\xi)|^2 d\xi \right]^{2/q} \geq \left[ \int_{S^n} |g(\xi)|^2 d\xi \right]^{2/q} = \left[ \int_{S^n} |f(\xi)|^2 d\xi \right]^{2/q}
\end{aligned}$$

where  $u_{\#}$  = harmonic extension of  $g^*(\xi)$  to the interior of the unit ball and using Dirichlet's principle

$$\int_{|x| \leq 1} |\nabla h|^2 dx \geq \int_{|x| \leq 1} |\nabla u_{\#}|^2 dx$$

for any  $h$  which is a smooth extension of  $g^*(\xi)$  to the interior of the unit ball, and

$$g^*(\xi) = \sup_{r>0} |g(r, \xi)| \geq |g(\xi)| = |f(\xi)|.$$

Hence putting all the steps together, inequality (15) obtained by using the dual-spectral form of the Hardy-Littlewood-Sobolev inequality on the sphere  $S^n$  (see page 233 in [1]) together with symmetrization on the multiplicative group  $\mathbb{R}_+$  results in a second derivation of inequality (14):

$$\begin{aligned}
\left[ \int_{S^n} |f(\xi)|^q d\xi \right]^{2/q} &\leq b_n \int_{\mathbb{R}^{n+1}} |\nabla f|^2 dx \\
b_n &= \frac{1}{4} \pi^{-(n+1)/2} \Gamma\left(\frac{n-1}{2}\right).
\end{aligned}$$

Still Theorem 4, from which inequality (14) is obtained, is a more general result as it includes fractional smoothing, and by explicit symmetric extension on  $\mathbb{R}_+$  can be used to obtain inequality (15) for harmonic extension on the unit ball in  $\mathbb{R}^{n+1}$ .

### 3. Proof of the reverse Hardy-Littlewood-Sobolev inequality.

The conformal invariant structure of the Hardy-Littlewood-Sobolev inequality can be continued across the Lebesgue index  $p = 1$  where for non-negative functions the inequality reverses.

**Theorem 16** (reverse Hardy-Littlewood-Sobolev inequality). *Let  $f, g \in L^p(\mathbb{R}^n)$  with  $f, g \geq 0$ ,  $0 < p < 1$  and  $\lambda = -2n/p'$  ( $p = 2n/(2n + \lambda)$ ); then*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) |x - y|^\lambda g(y) dx dy \geq A_\lambda \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} \quad (53)$$

$$A_\lambda = \pi^{\lambda/2} \frac{\Gamma(\frac{n+\lambda}{2})}{\Gamma(n + \frac{\lambda}{2})} \left[ \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right]^{1+\lambda/n}$$

with extremal functions given up to conformal automorphism by  $A(1 + |x|^2)^{-n/p}$ .

*Proof.* Since  $|x|^\lambda$  is a radial increasing function, apply symmetrization to obtain

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) |x - y|^\lambda g(y) dx dy \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} f^*(x) |x - y|^\lambda g^*(y) dx dy$$

where  $f^*, g^*$  denote the equimeasurable radial decreasing rearrangements of  $f, g$ . The next step is to reduce the problem to the multiplicative group  $\mathbb{R}_+$  or equivalently the line  $\mathbb{R}$ . Set  $h(u) = |x|^{n/p} f^*(x)$ ,  $k(v) = |y|^{n/p} g^*(y)$  where  $u = |x|$ ,  $v = |y|$ ; then the inequality becomes

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} h(u)k(v) \int_{S^{n-1} \times S^{n-1}} \left[ \frac{u}{v} + \frac{v}{u} - 2\xi \cdot \eta \right]^{\lambda/2} d\xi d\eta \frac{du}{u} \frac{dv}{v} \geq B_\lambda \|h\|_{L^p(\mathbb{R}_+)} \|k\|_{L^p(\mathbb{R}_+)} . \quad (54)$$

Observe that  $h, k$  are bounded so that  $h, k \in L^p(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ . The “potential” is now symmetric increasing away from the origin  $\{u = 1\}$  on  $\mathbb{R}_+$  so symmetrization will improve the inequality by diminishing the left-hand side so that  $h(1/u) = h(u)$  is monotone decreasing for  $u > 1$  (similarly for  $k$ ). This step then implies that inequality (53) is improved if (1)  $f$  is radial decreasing, (2)  $|x|^{n/p} f(x)$  is decreasing for  $|x| > 1$ , and (3)  $f(|x|^{-1}) = |x|^{2n/p} f(|x|)$ , all for nonnegative  $f$ . These conditions are precisely what is meant by saying that  $f$  and  $g$  possess “inversion symmetry”. Set  $u = e^t$  and  $v = e^s$  so that the working inequality becomes

$$\int_{\mathbb{R} \times \mathbb{R}} h(t)k(s) \int_{S^{n-1} \times S^{n-1}} [\cosh(t-s) - \xi \cdot \eta]^{\lambda/2} d\xi d\eta dt ds \geq C \|h\|_{L^p(\mathbb{R})} \|k\|_{L^p(\mathbb{R})} \quad (55)$$

Normalize this expression by setting  $\|h\|_{L^p(\mathbb{R})} = \|k\|_{L^p(\mathbb{R})} = 1$ ; let

$$\int_{S^{n-1} \times S^{n-1}} [\cosh(t) - \xi \cdot \eta]^{\lambda/2} d\xi d\eta = J_N(t) + K_N(t)$$

where  $J_N$  is supported on  $\{|t| < N\}$  and  $K_N$  is supported on  $\{|t| \geq N\}$ . First, show that for  $\|h\|_p = \|k\|_p = 1$

$$\inf \left[ \int_{\mathbb{R} \times \mathbb{R}} h(t)k(s) \int_{S^{n-1} \times S^{n-1}} [\cosh(t-s) - \xi \cdot \eta]^{\lambda/2} d\xi d\eta dt ds \right] = C > 0 .$$

This fact follows from the reverse Young’s inequality where

$$\int_{\mathbb{R} \times \mathbb{R}} h(t)k(s)K_1(t-s) dt ds \geq \|K_1\|_{L^{p'/2}(\mathbb{R})} > 0$$

so the positive infimum  $C$  exists. The second objective is to show the existence of extremals where the infimum is attained. Consider sequences  $\{h_m\}, \{k_m\}$  with  $\|h_m\|_p = \|k_m\|_p = 1$  so that

$$\Lambda_m = \int_{\mathbb{R} \times \mathbb{R}} h_m(t)k_m(s) \int_{S^{n-1} \times S^{n-1}} [\cosh(t-s) - \xi \cdot \eta]^{\lambda/2} d\xi d\eta dt ds \xrightarrow{m \rightarrow \infty} C$$

and  $2C > \lambda_m \geq C$ ; the functions  $h_m, k_m$  will be symmetric decreasing and uniformly bounded by a multiple of  $(1 + |t|)^{-1/p}$  where  $0 < p < 1$  so that they have a uniform  $L^1(\mathbb{R})$  majorant. Since the functions are decreasing, the Helly selection principle can be applied to choose subsequences that converge almost everywhere to functions  $h$  and  $k$  with  $\|h\|_p \leq 1, \|k\|_p \leq 1$ . To simplify notation, the pointwise convergent sequences are now substituted in place of the original sequences. By Fatou’s lemma

$$\Lambda_x = \int_{\mathbb{R} \times \mathbb{R}} h(t)k(s) \int_{S^{n-1} \times S^{n-1}} [\cosh(t-s) - \xi \cdot \eta]^{\lambda/2} d\xi d\eta dt ds \leq \lim \Lambda_m = C$$

Past arguments have used a uniform majorant for the sequential functions to show that limit and integral can be interchanged for the “potential functional” (see discussion on page 40 in [2], and the proof of Theorem 15 on the multilinear Hardy-Littlewood-Sobolev inequality in [10]). A contrasting strategy is utilized here where control by the “potential functional” combined with the monotonicity of the functions and the uniform  $L^1$  majorants will show that

$$\int_{\mathbb{R}} h_m(t) e^{\lambda/2|t|} dt < D_1 , \quad h_m(t) \leq D_2 e^{-\lambda/2|t|}$$

and similarly for  $\{k_m\}$  which gives

$$1 = \lim \int |h_m|^p dt = \int |h|^p dt, \quad 1 = \lim \int |k_m|^p dt = \int |k|^p dt.$$

Since  $h$  and  $k$  have unit norms,  $\Lambda_* = C = \text{infimum}$  and  $h, k$  are extremal functions for the reverse Hardy-Littlewood-Sobolev inequality. In returning to the  $\mathbb{R}^n$  setting, there will be extremal functions  $f, g$  with inversion symmetry. The conformal structure of the Hardy-Littlewood-Sobolev functional will determine allowed forms for the extremal functions from which the constant  $A_\lambda$  in equation (53) can be calculated. Since the functional is bilinear, the most direct approach is use conformal symmetry on the sphere  $S^n$ . For  $\xi, \eta \in S^n$ , let  $f(x) = (1 + |x|^2)^{-n/p} F(\xi)$  and  $g(y) = (1 + |y|^2)^{-n/p} G(\eta)$ ; then

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^\lambda f(x) g(y) dx dy = C_\lambda \int_{S^n \times S^n} |\xi - \eta|^\lambda F(\xi) G(\eta) d\xi d\eta$$

$$C_\lambda = 2^{-\lambda} \pi^n \left[ \Gamma(n/2) / \Gamma(n) \right]^2$$

where  $d\xi, d\eta$  denote normalized surface measure on  $S^n$  with the map from  $\mathbb{R}^n$  to  $S^n$  defined by

$$\xi = \left( \frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right), \quad d\xi = \pi^{-n/2} \left[ \Gamma(n) / \Gamma(n/2) \right] (1 + |x|^2)^{-n} dx$$

$$|x - y| = \frac{1}{2} |\xi - \eta| \left[ (1 + |x|^2)(1 + |y|^2) \right]^{1/2}$$

Then inequality (53) has an equivalent formulation on the  $n$ -dimensional sphere:

$$\int_{S^n \times S^n} F(\xi) |\xi - \eta|^\lambda G(\eta) d\xi d\eta \geq B_\lambda \|F\|_{L^p(S^n)} \|G\|_{L^p(S^n)} \quad (56)$$

$$B_\lambda = \int_{S^n} |\xi - \eta|^\lambda d\xi = 2^\lambda \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n+\lambda}{2})}{\Gamma(n + \frac{\lambda}{2})}$$

Because only two functions are involved, two-point symmetrization can be used to show that the inequality must be improved by rearranged functions that depend only on the polar angle and are decreasing away from a pole. But the inequality cannot be improved so the extremal functions at this stage must combine two properties: a) monotone decreasing away from a pole; b) possess “inversion symmetry” which on the sphere means that functions are symmetric with respect to an equator. Then up to conformal automorphism, the only possible extremals on  $S^n$  are constant. And this remark completes the proof of Theorem 16. For the dimension  $n$  at least two, an alternative determination of the form of the extremals can be obtained using the hyperbolic symmetry of the Hardy-Littlewood-Sobolev functional. For  $2 \leq \ell \leq n$  with the Poincaré distance and left-invariant Haar measure on  $\mathbb{H}^\ell$  for  $w = (x, y) \in \mathbb{R}^{\ell-1} \times \mathbb{R}_+$

$$d(w, w') = \frac{|w - w'|}{2\sqrt{yy'}}, \quad d\nu = y^{-\ell} dy dx$$

an inequality equivalent to (53) is given by

$$\int_{\mathbb{H}^\ell \times \mathbb{H}^\ell} F(w) G(w') \int_{S^{n-\ell} \times S^{n-\ell}} \left[ d^2(w, w') + (1 - \xi \cdot \xi')/2 \right]^{\lambda/2} d\xi d\xi' d\nu d\nu' \quad (57)$$

$$\geq D_\lambda \|F\|_{L^p(\mathbb{H}^\ell)} \|G\|_{L^p(\mathbb{H}^\ell)}$$

The constraint of possessing radial symmetry on  $\mathbb{R}^n$  in (53) and geodesic radial symmetry on  $\mathbb{H}^\ell$  in (57) will determine the form of the extremals (see the argument given in [3] concerning “axial symmetry and  $SL(2, R)$ ” and the proof of Theorem 15 in [10] for the multilinear Hardy-Littlewood-Sobolev inequality.  $\square$

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